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Numerical Solutions For Singularly Perturbed Differential-Difference Equations: A Dual-Layer Fitted Approach

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Abstract

Singularly perturbed differential-difference equations (SPDDE) pose significant challenges due to their oscillatory behavior and unsatisfactory results when traditional numerical methods are applied with large step sizes relative to the perturbation parameter ε. In this study, we propose a novel computational approach for solving SPDDE, leveraging a dual-layer fitted method and Taylor series expansion. First, the given SPDDE is reduced to an ordinary singularly perturbed problem using Taylor series expansion to handle terms involving negative and positive shifts. Subsequently, a three-term numerical scheme is derived using finite differences, augmented by a fitting factor derived from singular perturbation theory to enhance accuracy and stability. The resulting tridiagonal system of equations is efficiently solved using the Thomas algorithm. To validate the proposed method, we solve model problems with varying values of ε, delay parameter δ, and advance parameter η. The computational results are compared with existing literature, presenting maximum absolute errors and graphical representations. Our approach demonstrates significant improvements in accuracy and stability, making it a valuable tool for researchers tackling SPDDE in diverse applications.

Key words: *Accuracy, Model Issues, Dual-Layer Fitted Method, Positive shift, Negative Shift.*

1. Introduction

In the realm of control systems, the ubiquitous presence of time delays cannot be overlooked, stemming from the finite duration required for information sensing and subsequent response. This intrinsic quality gives rise to the development of singularly perturbed differential-difference equations (SPDDEs). Such equations encompass mean differential equations, featuring a minute positive parameter scaling the leading derivative, and include at least one shift term “such as a delay or advance”. This forms a crucial area of investigation in scientific and engineering fields, where a nuanced understanding of intricate dynamics is essential for advancing research and innovation.

The intrigue of SPDDEs lies in their multi-scale nature; they exhibit thin transition layers where solutions undergo rapid variation, while maintaining stability away from these layers, where variations occur at a slower pace. This intricate behavior renders SPDDEs fundamental to theoretical explorations and practical applications across various fields, including control theory (M.W. Derstineet al.,1982), physiology (K. Ikeda et al.,1982) and neural networks(M.K. Kadalbajoo, K.K. Sharma, 2005), among others.

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Previous research efforts have delved into the complexities of SPDDEs, utilizing diverse techniques to unravel their intricate structure. Literature has explored approximate solutions employing methods such as matched asymptotic expansions and Laplace transforms, facilitating an in-depth understanding of the layer structures inherent in these differential-difference equations. Moreover, computational advancements have enabled the numerical simulation of SPDDEs, allowing researchers to explore their behavior under various conditions and parameter settings. These simulations provide valuable insights into the dynamics of SPDDEs and aid in the validation of theoretical findings. The study of SPDDEs has spurred interdisciplinary collaborations, drawing expertise from mathematics, physics, engineering, and beyond. Such collaborations foster a comprehensive approach to understanding SPDDEs and harnessing their potential in diverse applications. In essence, the allure of SPDDEs lies not only in their mathematical intricacies but also in their profound implications for real-world phenomena, making them a focal point of ongoing research and innovation.

Researchers have also proposed innovative numerical integration techniques, ranging from Numerov's difference scheme to exponential fitted methods, offering robust solutions tailored to specific types of SPDDEs. Furthermore, advancements in parametric spline schemes and fitted finite difference methods have expanded the toolkit for addressing nonlinear SPDDEs, providing a comprehensive approach to deciphering and solving these complex problems.This introduction sets the stage for a comprehensive exploration of SPDDEs, highlighting their significance, prevalence in real-world scenarios, and the diverse methodologies employed to comprehend and solve them. Through this study, we delve into the intricate world of SPDDEs, aiming to contribute to the existing body of knowledge and enhance our understanding of these intriguing equations.

2. Method Description

Considering SPDDE of the form:

$$\varepsilon^2 u''(t) + b(t)u(t-\delta) + c(t)u(t) + d(t)u(t+\eta) = f(t) \tag{1}$$

$\forall t \in (0,1)$ subject to the interval and boundary conditions

$$u(t) = \varphi(t) \ \text{ on } \ -\delta \le t \le 0 \tag{2}$$

$$u(t) = \gamma(t) \ \text{ on } \ 1 \le t \le 1+\eta \tag{3}$$

where

$a(t), b(t), c(t), d(t), f(t), \varphi(t)$ and $\gamma(t)$are sufficiently smooth functions on (0, 1), $0< \varepsilon < < 1$ is the perturbation parameter and $0 < \delta = O(\varepsilon)$ and$0 < \eta = O(\varepsilon)$ are the delay (negative shift) and the advance(positive shift) parameters respectively.

In the neighbourhood of point t, Taylor's Expansion is used and we get

$$u(t-\delta) \approx u(t) - \delta u'(t) \tag{4}$$

$$u(t+\eta) \approx u(t) + \eta u'(t) \tag{5}$$

Using Equation (4) and Equation (5) in Equation (1), we get an asymptotically equivalent singularly perturbed boundary value problem of the form:

$$\varepsilon^2 u''(t) + p(t)u'(t) + q(t)u(t) = f(t) \tag{6}$$

$$u(0) = \varphi(0) = \varphi_o \tag{7}$$

$$u(1) = \gamma(1) = \gamma_1 \tag{8}$$

where

$$p(t) = d(t)\eta - b(t)\delta \tag{9}$$

$$q(t) = b(t) + c(t) + d(t) \tag{10}$$

Since $0 < \delta << 1$and $0 < \eta << 1$, the transition from Equation (1) to Equation (6) is admissible. Further details on the validity of this transition are found in El'sgol'ts and Norkin. If $q(t) \leq 0$ on the interval [0,1], then the solution of Equation (1) exhibits boundary layers at each edge of the interval [0,1], while it exhibits oscillatory behavior for $q(t) > 0$. Now, we consider dual layer problems.

From the singular perturbations, the solution of Equations (6) -(8) is of the form

$$u(t) = u_o(x) + \frac{p(o)}{p(t)}(\varphi_o - u_o(o))e^{-\int_o^t\left(\frac{p(t)}{\varepsilon^2} - \frac{q(t)}{p(t)}\right)dx} + O(\varepsilon) \quad (11)$$

Where,$u_o(t)$ is the solution of -

$$p(t)u_o'(t) + q(t)u_o(t) = f(t),\ u_o(1) = \gamma_1 \quad (12)$$

Using the Taylor's series expansion for p(t) and q(t) about the point '$t = 0$' and limiting to their first terms, Equation (11) becomes,

$$u(t) = u_o(t) + (\varphi_o - u_o(0))e^{-\left(\frac{p(o)}{\varepsilon^2} - \frac{q(o)}{p(o)}\right)t} + O(\varepsilon) \quad (13)$$

on discretization of the interval [0,1] into N equal subintervals of step size $h = \frac{1}{N}$ to make sure that $t_i = ih$, i =0, 1, 2,.… ,N..

From Equation (13), we have

$$u(ih) = u_o(ih) + (\varphi_o - u_o(0))e^{-\left(\frac{p(o)}{\varepsilon^2} - \frac{q(o)}{p(o)}\right)ih} + O(\varepsilon)$$

Therefore

$$\lim_{h\to o} u(ih) = u_o(0) + (\varphi_o - u_o(o))e^{-\left(\frac{p^2(o) - \varepsilon q(o)}{p(o)}\right)i\rho} \quad (14)$$

where $\rho = \frac{h}{\varepsilon^2}$

Supposing that $u(t)$ is continuously differentiable in the interval [0,1] and applying Taylor's series expansion for $u(t_{i+1})$ and $u(t_{i-1})$, we have:

$$u(t_{i+1}) = u_{i+1} = u_i + hu_i' + \frac{h^2}{2!}u_i'' + \frac{h^3}{3!}u_i''' + \frac{h^4}{4!}u_i^{(4)} + \frac{h^5}{5!}u_i^{(5)} + \frac{h^6}{6!}u_i^{(6)} + \frac{h^7}{7!}u_i^{(7)} + \frac{h^8}{8!}u_i^{(8)} + O(h^9)$$

$$u(t_{i-1}) = u_{i-1} = u_i - hu_i' + \frac{h^2}{2!}u_i'' - \frac{h^3}{3!}u_i''' + \frac{h^4}{4!}u_i^{(4)} - \frac{h^5}{5!}u_i^{(5)} + \frac{h^6}{6!}u_i^{(6)} - \frac{h^7}{7!}u_i^{(7)} + \frac{h^8}{8!}u_i^{(8)} - O(h^9)$$

From the finite differences, we have

$$u_{i-1} - 2u_i + u_{i+1} = \frac{2h^2}{2!}u_i'' + \frac{2h^4}{4!}u_i^{(4)} + \frac{2h^6}{6!}u_i^{(6)} + \frac{2h^8}{8!}u_i^{(8)} + O(h^{10}) \quad (15)$$

Now we have the relation:

$$u_{i-1}'' - 2u_i'' + u_{i+1}'' = \frac{2h^2}{2!}u_i^{(4)} + \frac{2h^4}{4!}u_i^{(6)} + \frac{2h^6}{6!}u_i^{(8)} + \frac{2h^8}{8!}u_i^{(10)} + O(h^{12})$$

Substituting $\frac{h^4}{12}u_i^{(6)}$ from the above equation in Eq.(15), we have

$$u_{i-1} - 2u_i + u_{i+1} = h^2u_i'' + \frac{h^4}{12}u_i^{(4)} + \frac{h^2}{30}\left[u_{i-1}'' - 2u_i'' + u_{i+1}'' - h^2u_i^{(4)} - \frac{h^6}{360}u_i^{(8)}\right] + \frac{2h^8}{8!} + O(h^{10})$$

$$u_{i-1} - 2u_i + u_{i+1} = h^2\left[u_i'' + \frac{1}{30}(u_{i-1}'' - 2u_i'' + u_{i+1}'')\right] + \frac{h^4}{12}u_i^{(4)} - \frac{h^4}{30}u_i^{(4)} - \frac{h^6}{10800}u_i^{(8)} + \frac{2h^8}{8!}u_i^{(8)} + O(h^{10})$$

and

$$u_{i-1} - 2u_i + u_{i+1} = \frac{h^2}{30}(u_{i-1}'' + 28u_i'' + u_{i+1}'') + R, \quad (16)$$

Where $R = \frac{h^4}{20} u_i^{(4)} + \frac{13h^6}{302400} u_i^{(8)} + O(h^{10})$

Now from the Equation (6), we have

$$\varepsilon u''_{i+1} = -p_{i+1}u'_{i+1} - q_{i+1}u_{i+1} + f_{i+1}$$
$$\varepsilon u''_i = -p_i u'_i - q_i u_i + f_i$$
$$\varepsilon u''_{i-1} = -p_{i-1}u'_{i-1} - q_{i-1}u_{i-1} + f_{i-1} \quad (17)$$

Utilizing the subsequent three-point approximations for first-order derivatives:

$$u'_{i+1} \simeq \frac{u_{i-1} - 4u_i + 3u_{i+1}}{2h}$$
$$u'_i \simeq \frac{u_{i+1} - u_{i-1}}{2h}$$
$$u'_{i-1} \simeq \frac{-3u_{i-1} + 4u_i - u_{i+1}}{2h} \quad (18)$$

Substituting Equation (17) and Equation (18) in Equation (16) and condensing we get

$$\varepsilon\left(\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2}\right) + \frac{p_{i-1}}{60h}[-3u_{i-1} + 4u_i - u_{i+1}] + \frac{28p_i}{60h}[u_{i+1} - u_{i-1}] + \frac{p_{i+1}}{60h}[u_{i-1} - 4u_i + 3u_{i+1}] + \frac{q_{i-1}}{30}y_{i-1} + \frac{28q_i}{30}y_i + \frac{q_{i+1}}{30}y_{i+1} = \frac{[f_{i-1} + 28f_i + f_{i+1}]}{30} \quad (19)$$

The tridiagonal system Eq. (19) is given by

$A_i u_{i-1} - B_i u_i + C_i u_{i+1} = D_i$,

for i = 1,2,…,N-1 (20)

where

$$A_i = \frac{\varepsilon}{h^2} - \frac{3p_{i-1}}{60h} + \frac{q_{i-1}}{30} - \frac{28p_i}{60h} + \frac{p_{i+1}}{60h}$$
$$B_i = \frac{2\varepsilon}{h^2} - \frac{4p_{i-1}}{60h} - \frac{28q_i}{30} + \frac{4p_{i+1}}{60h}$$
$$C_i = \frac{\varepsilon}{h^2} - \frac{p_{i-1}}{60h} + \frac{q_{i+1}}{30} + \frac{28p_i}{60h} + \frac{3p_{i+1}}{60h}$$
$$D_i = \frac{1}{30}[f_{i-1} + 28f_i + f_{i+1}]$$

To solve the tidiagonal system of Equation (20), Thomas algorithm is used.

3. Numerical examples

The suggested approach is verified using examples featuring equations akin to Equations (1) through (3). This validation process extends to a SPDDE.

$$\varepsilon^2 u''(x) + b(x)u(x-\delta) + c(x)u(x) + d(x)u(x+\eta) = f(x)$$

$\forall x \in (0,1)$& subject to the interval and boundary conditions

$$u(x) = \varphi(x), \qquad on -\delta \le x \le 0$$
$$u(x) = \gamma(x), \qquad on \;\; 1 \le x \le 1+\eta$$

with constant coefficients(i.e,

$b(x) = b, c(x) = c, d(x) = d, f(x) = f, \varphi(x) = \varphi \; and \; \gamma(x) = \gamma)$

is given by $u(x) = \frac{[(1-b-c-d)e^{m_2} - 1]e^{m_1 x} - [(1-b-c-d)e^{m_1} - 1]e^{m_2 x}}{(b+c+d)(e^{m_1} - e^{m_2})} + \frac{1}{b+c+d}$

where

$$m_1 = \frac{(b\delta - d\eta) + \sqrt{(d\eta - b\delta)^2 - 4\varepsilon^2(b+c+d)}}{2\varepsilon^2}, \qquad m_2 = \frac{(b\delta - d\eta) - \sqrt{(d\eta - b\delta)^2 - 4\varepsilon^2(b+c+d)}}{2\varepsilon^2}$$

Example 1. Considering the "Singularly perturbed partial differential equation”& constant coefficients together:

$$\varepsilon^2 u''(x) - 2u(x-\delta) - u(x) - 2u(x+\eta) = 1\,, \quad \varphi(x) = 1, \qquad \gamma(x) = 0$$

Table 1 and 2 & Figure 1 and 2 represents the results observed.

Example 2. Considering the "Singularly perturbed partial differential equation”& constant coefficients together:

$$\varepsilon^2 u''(x) + 0.25u(x-\delta) - u(x) + 0.25u(x+\eta) = 1\,, \quad \varphi(x) = 1, \qquad \gamma(x) = 0$$

Table 3 and 4 & Figure 3 and 4 represents the results observed.

Table 1: In solution of Example 1, "the numerical results for N=100, $\varepsilon = 0.1$ and $\delta = 0.07$"

x	$\eta = 0$		$\eta = 0.03$		$\eta = 0.06$	
	Num. Sol.	Exact Sol.	Num. Sol.	Exact Sol.	Num. Sol.	Exact Sol.
0.00	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.02	0.45242458	0.45264934	0.50300731	0.50344253	0.55169184	0.55214441
0.04	0.15471487	0.15495931	0.21184940	0.21235949	0.27086719	0.27143434
0.06	-0.00714606	-0.00694668	0.04127762	0.04172600	0.09495586	0.09548892
0.08	-0.09514776	-0.09500320	-0.05865005	-0.05829970	-0.01523674	-0.01479139
0.10	-0.14299314	-0.14289488	-0.11719162	-0.11693498	-0.08426247	-0.08391365
0.20	-0.19729146	-0.19728211	-0.19428558	-0.19425011	-0.18883735	-0.18876996
0.40	-0.19998330	-0.19998323	-0.19997011	-0.19996975	-0.19989563	-0.19989436
0.60	-0.19971798	-0.19971742	-0.19988711	-0.19988642	-0.19996039	-0.19995998
0.80	-0.19248997	-0.19248250	-0.19525107	-0.19523670	-0.19721989	-0.19720588
0.90	-0.16124429	-0.16122502	-0.16918139	-0.16913482	-0.17641992	-0.17636058
0.92	-0.14618856	-0.14616716	-0.15520223	-0.15514807	-0.16383885	-0.16376607
0.94	-0.12528403	-0.12526175	-0.13488218	-0.13482316	-0.14454520	-0.14446151
0.96	-0.09625855	-0.09623793	-0.10534508	-0.10528788	-0.11495748	-0.11487195
0.98	-0.05595734	-0.05594302	-0.06241008	-0.06236852	-0.06958335	-0.06951778
1.00	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
"Maximum Error: 3.0469e-04"			"1.8305e-04"		"4.8258e-04"	

Table 2: In solution of Example 1, "the maximum absolute errors for $\delta = \varepsilon^2$ and $\eta = 2\varepsilon^2$"

$\varepsilon \backslash N$	128	256	512	1024	2048
0.1	3.6651e-04	9.1836e-05	2.2972e-05	5.7438e-06	1.4360e-06
0.01	2.1734e-02	7.7833e-03	2.0641e-03	5.2711e-04	1.3272e-04

Table 3: In solution of Example 2, "the numerical results for N=100, $\varepsilon = 0.01$ and $\eta = 0.007$"

x	$\delta = 0$		$\delta = 0.003$		$\delta = 0.006$	
	Num. Sol.	Exact Sol.	Num. Sol.	Exact Sol.	Num. Sol.	Exact Sol.

0.00	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.02	-1.38767494	-1.39431045	-1.33302029	-1.34238288	-1.27719916	-1.28881467
0.04	-1.87501934	-1.87771339	-1.85171269	-1.85584657	-1.82585298	-1.83140514
0.06	-1.97449040	-1.97531076	-1.96703179	-1.96840075	-1.95804213	-1.96003260
0.08	-1.99479328	-1.99501533	-1.99267029	-1.99307326	-1.98989094	-1.99052526
0.10	-1.99893726	-1.99899361	-1.99837041	-1.99848162	-1.99756439	-1.99775390
0.20	-1.99999962	-1.99999966	-1.99999911	-1.99999923	-1.99999802	-1.99999832
0.40	-2.00000000	-2.00000000	-2.00000000	-2.00000000	-2.00000000	-2.00000000
0.60	-2.00000000	-2.00000000	-2.00000000	-2.00000000	-2.00000000	-2.00000000
0.80	-1.99999110	-1.99999255	-1.99999548	-1.99999621	-1.99999781	-1.99999815
0.90	-1.99578031	-1.99613909	-1.99699376	-1.99724841	-1.99790619	-1.99807730
0.92	-1.98553127	-1.98652411	-1.98896875	-1.98972284	-1.99174047	-1.99228502
0.94	-1.95038873	-1.95296451	-1.95952135	-1.96161494	-1.96741828	-1.96904301
0.96	-1.82988986	-1.83583000	-1.85146549	-1.85663224	-1.87147350	-1.87578258
0.98	-1.41671595	-1.42699041	-1.45495961	-1.46452309	-1.49299605	-1.50156761
1.00	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
"Maximum Error: 1.0274e-02"			"9.9633e-03"		"1.1880e-02"	

Table 4: In solution of Example 2, "the maximum absolute errors for $\delta = \varepsilon^2$ and $\eta = 2\varepsilon^2$"

ε\N	128	256	512	1024	2048
0.1	8.2492e-05	2.0630e-05	5.1581e-06	1.2896e-06	3.2239e-07
0.01	8.0762e-03	2.0682e-03	5.2272e-04	1.3088e-04	3.2750e-05

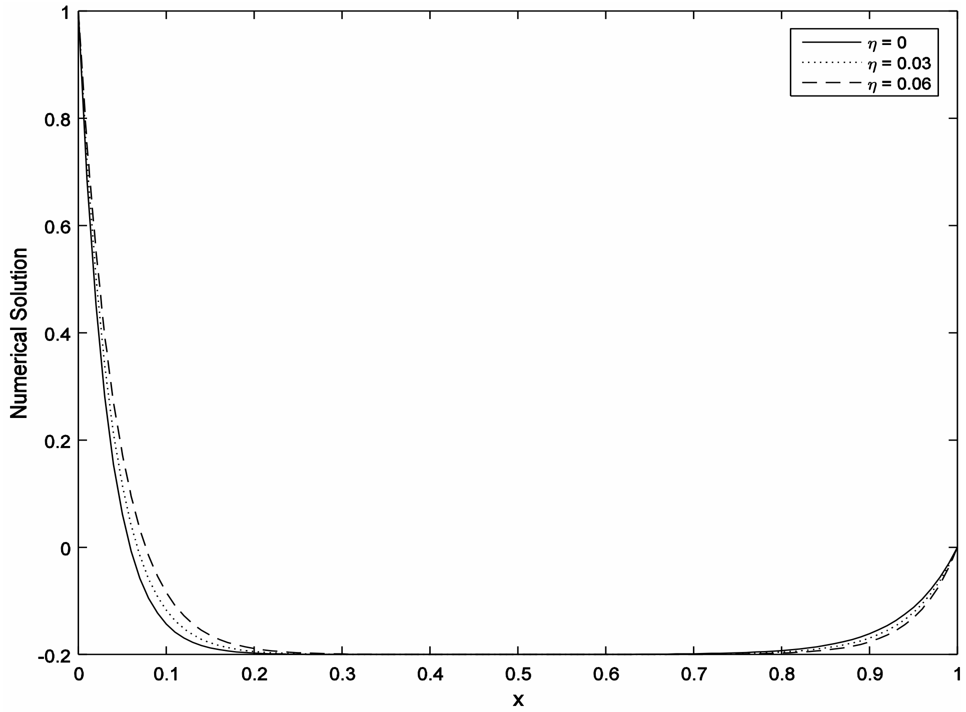

Figure 1. Numerical solution of Example 1 for "N=100, $\varepsilon = 0.1$ and $\delta = 0.07$"

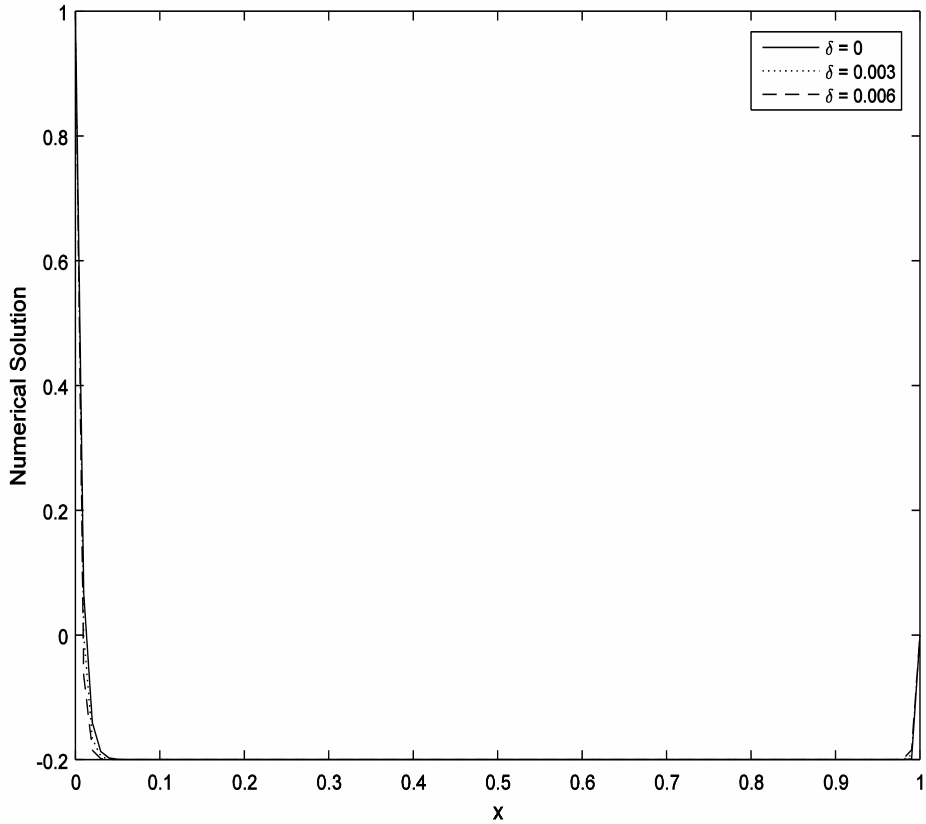

Figure 2. Numerical solution of Example 1 for "N=100, $\varepsilon = 0.01$ and $\eta = 0.007$"

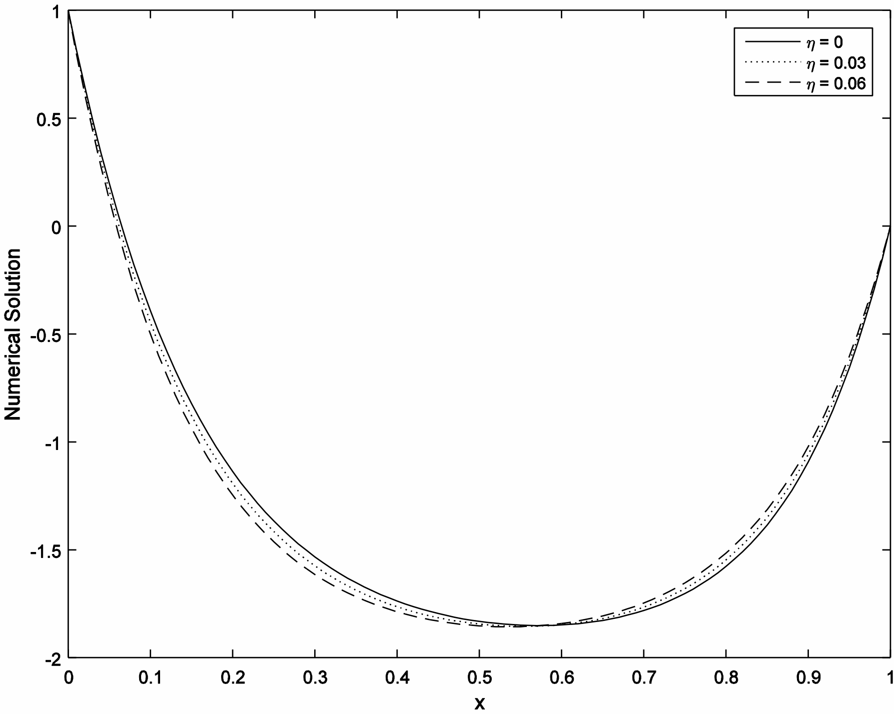

Figure 3. Numerical solution of Example 2 for "N=100, $\varepsilon = 0.1$, and $\delta = 0.07$"

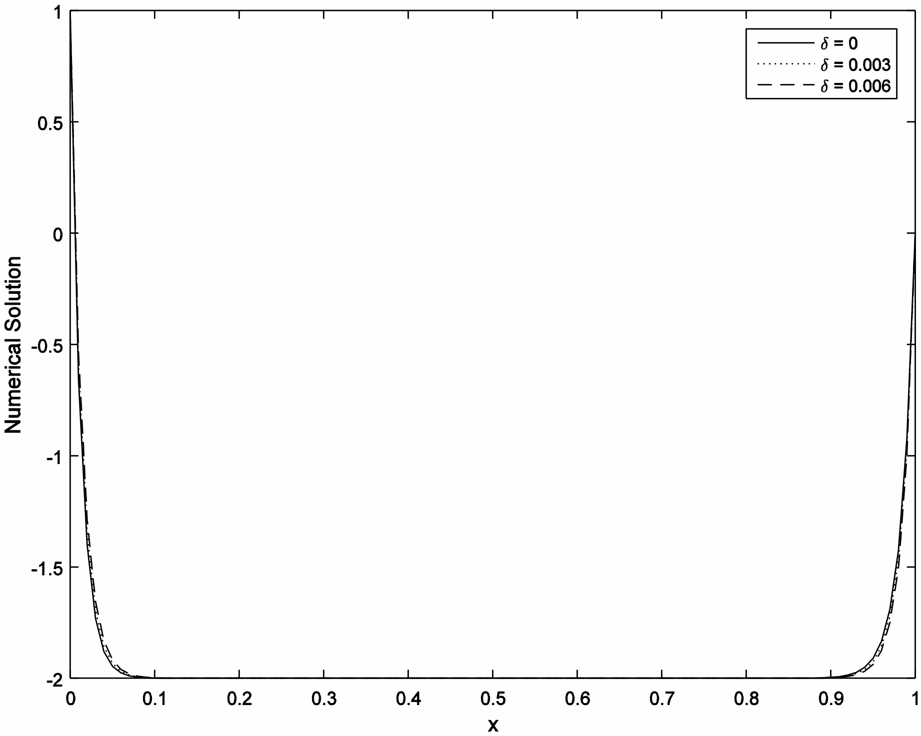

Figure 4. Numerical Solution of Example 2 for "N=100, $\varepsilon = 0.01$, and $\eta = 0.007$"

4. Discussions and conclusion

This study has introduced and applied a tailored fitted method for solving singularly perturbed differential-difference equations displaying dual-layer behavior. Through a series of model problems involving variations in parameters such as ε, δ, η, and h, we systematically evaluated the performance of our method. By presenting the maximum absolute errors and computational orders for well-established examples from the literature, we conducted a comprehensive analysis. The comparison between our numerical solutions and exact solutions validated the accuracy and reliability of our proposed approach. The results unequivocally show that our method excels in approximating exact solutions, affirming its robustness and effectiveness in dealing with the intricacies of singularly perturbed differential-difference equations featuring dual-layer phenomena. This study underscores the practical applicability and potential of the fitted method in accurately capturing the behavior of complex systems governed by such equations.

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