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Brief History of Fractional Calculus: A Survey

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Abstract: During the last several decades, there seems to have taken place a revival of interest in fractional calculus due to its vast applications in various fields. In this survey paper a brief history of fractional calculus is discussed with step-wise development of fractional derivatives, their application and importance in various fields.
Keywords: Fractional Calculus, Fractional Derivatives, Arbitrary order, Reimann-Liouville (R-L) Fractional Integral, Cauchy Integral Formula.

Introduction:

Fractional calculus may be known to be the extension of the concept of a derivative operator from integer order to fractional/arbitrary order [5]. Concept of fractional calculus was discussed as soon as the idea of classical calculus developed. A communication between L' Hôpital and Gottfried Wilhelm Leibniz in 1695, L' Hôpital ask to Leibniz what if we put $n = \frac{1}{2}$, in $\frac{d^n}{dx^n}$. Fractional calculus evolves slowly but from last several decades researchers realize, its importance and vast application in various fields. Therefore, now day's fractional calculus is at the center of research for scientists, engineers and mathematicians.

History of Fractional Calculus:

Law of indices given by Joseph-Louis Lagrange (1722) for differential operators play indirect role to understand fractional operators, because law of indices doesn't work in most of the fractional derivative's case. Laplace (1812) was the first mathematician who defined fractional derivative with the help of integral base approach, but it was Lacroix who first give the fractional derivative of arbitrary order for $\frac{d^n x^m}{d^n}$; for $m = 1 \; and \; n = \frac{1}{2}$,

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} x = \frac{2\sqrt{x}}{\sqrt{\pi}} . \tag{1}$$

Joseph B. J. Fourier (1822) was the first mathematician that proposed the fractional derivative of trigonometric function. He defined the fractional derivative of arbitrary order of trigonometric functions as follows

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(r)\, dr \int_{-\infty}^{\infty} \cos w(x-r)\, dw \quad (2)$$

$$\frac{d^n}{d\,x^n} \cos w(x-r) = w^n \cos[w(x-r) + \frac{1}{2} n\pi] \quad (3)$$

Similarly, fractional derivative of arbitrary order (α) with the help of Fourier transform is

$$\frac{d^\alpha}{d\,x^\alpha} f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(r)\, dr \int_{-\infty}^{\infty} w^\alpha \cos\left[w(x-r) + \frac{1}{2}\alpha\pi\right] dw \quad (4)$$

where α will be interpreted as any number, whether positive or negative.

Niels Henrik Abel (1823) was the first mathematician who used fractional calculus to solve an integral equation (Tautochrone problem) that served as a single example of the use of fractional calculus for a very long time [5]. The integral he worked with is

$$\int_0^x (x-t)^{-\frac{1}{2}} f(t) dt \quad (5)$$

The first major study in the field of fractional calculus was led by Liouvill**e** (1832) based on the Fourier fractional integral and Abel solution on potential theory. Fractional derivative of exponential functions was given by Liouville and he extended the result of ordinary derivative to fractional derivative in natural way on some limited functions.

The derivative of integer order,

$$D^m e^{sx} = s^m e^{sx}$$

This result was naturally extended by Liouville to the derivative of arbitrary/fractional order.

$$D^\alpha e^{sx} = s^\alpha e^{sx}$$

His working assumption was that a function $f(x)$ could have any arbitrary derivative that could be expanded in series form.

$$f(x) = \sum_{m=0}^{\infty} a_m e^{s_m x}, \quad Re(s_m) > 0$$

is $$D^\alpha f(x) = \sum_{m=0}^{\infty} a_m s_m^\alpha e^{s_m x} \quad (6)$$

Expression (6) is known as Liouville's first definition of fractional derivative [2]. Later he gave another definition of fractional derivative based on integral approach.

$$T = \int_0^\infty s^{b-1} e^{-ys} ds, \; b > 0, \; y > 0.$$

Put $ys = z$ then

$$T = y^{-b} \int_0^\infty z^{b-1} e^{-z} dz$$

$$T = y^{-b} \Gamma(b)$$

$$y^{-b} = \frac{1}{\Gamma(b)} \; T$$

$$D^{\alpha}\, y^{-b} = \frac{(-1)^{\alpha}}{\Gamma(b)} \int_0^{\infty} s^{b+\alpha-1} \; e^{-ys} \; ds$$

Liouville's second definition

$$D^{\alpha}\, y^{-b} = \frac{(-1)^{\alpha}\, \Gamma(b+\alpha)}{\Gamma(b)}\, y^{-b-\alpha}, \qquad b > 0 \tag{7}$$

Liouville was the first mathematician who attempted to solve differential equations with the help of fractional operators, but due to his idea of complementary function in fractional differential equation, overall theory of fractional operators was viewed with distrust. Using the Lacroix-Peacock method, William Center discovered that the fractional derivative of a constant function is not equal to zero [2]. According to equation (7) Liouville fractional derivative of constant function is zero. G.F.B. Riemann gave the idea of fractional operator with the help of fractional integration based on the Taylor series

$$D^{-\alpha}[f(x)] = \frac{1}{\Gamma(\alpha)} \int_c^x (x-t)^{\alpha-1} f(t)dt + \Phi(x) \tag{8}$$

Due to ambiguity in lower limit c he added a complementary function $\Phi(x)$. Due to complementary function, law of exponent doesn't hold for Reimann fractional operator. In 1880 Cayley raised a question that due to presence of complementary function in Reimann fractional operator, it contains an infinity number of arbitrary constants. The fractional derivative for powers was studied systematically by Reimann [1847] but first attempts to solve this problem were already made by Leibniz and Euler. In 1869 N. Sonin published a paper titled "On differentiation with arbitrary index" which was based on the Cauchy's Integral formula. Later, four papers on this topic published by A.V. Letinkov "An explanation of the main concepts of the theory of differentiation of arbitrary index" which was gave extension to Sonin paper. Cauchy integral formula for integer is

$$D^m[f(z)] = \frac{m!}{2\pi i} \oint \frac{f(\xi)}{(\xi - z)^{n+1}} \; d\xi \tag{9}$$

Laurent suggests that we can generalizing the $m! = \Gamma(m+1)$, then, instead of a pole, the integrand contains a branch point. Laurent suggests that, in contrast to closed circuit, his contour is an open circuit on the Reimann surface. N. Sonin and A.V. Letinkov, which gave the new definition of fractional integral

$$D_x^{-\alpha}[f(x)] = \frac{1}{\Gamma(\alpha)} \int_c^x (x-\xi)^{\alpha-1} f(\xi)d\xi \;, \quad Re(\alpha) > 0 \tag{10}$$

This definition of fractional operators connects the theory of fractional calculus to the theory of generalized operators. When $x > c$ in (10) we have Reimann definition. The most used version occurs when $c = 0$.

$$D_x^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\xi)^{\alpha-1} f(\xi)d\xi \;, \quad Re(\alpha) > 0 \tag{11}$$

The form (11) referred as Reimann-Liouville fractional integral.
Reisz (1949) has developed a theory of fractional integration for function of more than one variable. There is an ambiguity in the definition of fractional derivative given by Reimann and Liouville (R-L). As per R-L, definition, fractional derivative of constant function is not zero. Caputo (1967) who noticed it and gave another simplified definition of fractional operator on the basis of series expansion. In contrast to Liouville definition of fractional derivative, if we apply the Caputo definition to exponential function then fractional derivative is not an exponential function but a more comprehensive Mittag-Leffler function. This is the primary disagreement in fractional calculus that has arisen, with various definitions of fractional operators being proposed, gives different result when applies on the same function. Under the sponsorship of "National Science Foundation, the first international conference on fractional calculus took place in 1974 in the University of New Haven, Connecticut". The theory of fractional derivatives progressed for three centuries as primarily a theoretical study of mathematics relevant only to mathematicians. In the 1980's, Mandelbrot works on fractional geometry drew the attention of physicists to this field of study and this led to the beginning of several publications in the field of fractional Brownian motion and anomalous diffusion processes [5]. In 1984, the second international conference on fractional calculus was held at Glasgow, Scotland sponsored by the University of Strathclyde. In 1989, the third international conference was held at Nihon University, Tokyo Japan [2]. Heaviside, Kelland, Bromwich and Harold's contributions in the field of fractional calculus are also valuable. Raspini (2000) deduced a symmetric wave equation, which turned out to be of fractional nature and this raised a question in fundamental physics which cannot be formulated using traditional methods [5]. So many mathematicians contributed in this field, few of them names are Al-Bassam, Erdelyi, T. Osler, M. Riesz, Kober, G.H. Hardy, H.T. Davis, I. Sneddon and recent one is Oldham and Spanier, A. Kilbas, H.M. Srivastava, R.K. Saxena, B. Ross, Igor Podlubny, Udita N. Katugampola, R. Khalil, Richard Herrmann and many more scientists. Physicists and engineers are still contributing in Engineering, Electrical Network, Signal Processing, Electromagnetic Theory and Probability [2].

Recent development and application in fractional calculus:

Most of the fractional derivatives that are defined previously are based on fractional integral approach which inherent local behavior and memory effect. Katugampola and R. Khalil introduced different definitions of fractional derivatives, which was based on integral and limit based approach respectively. After the limit based fractional derivative approach was introduced, some more definitions were introduced. For example, Conformable fractional derivative, Deformable fractional derivative etc.Due to the vast possibility of research and its application, in the last several years, we can see a huge amount of research in fractional operators, and not only in Mathematics and Engineering. Applications outside Mathematics include some topics as, Design of heat flux, Dissemination of atmospheric pollutants, Design of differentiator (Signal processing), Transmission line theory and Quantum mechanical calculations [1]. The other large field which requires the use of derivative of arbitrary order is the elaborated theory of fractals and the description of rheological properties of rocks [3]. Till the time no unique definition of fractional operators exists, even worse over time some additional definitions have been proposed.

So, Equations are unique if we use this fractional calculus method to solve fractional differential equations but will give different solutions based on the definition of fractional derivatives used. Therefore, engineers and physicists are in advantage position as compare to mathematicians. They have opportunities to compare theoretical results with experimental data [5].

Conclusion:

Fractional calculus has made an impact on all fields of classical analysis. It is clear that compared to the integer-order models, the recent fractional-order models are better and more effective. Non-local effects that may occur in space and time in high energy physics have developed as a result of physicists' growing interest in non-local field theories over the past several years. Fractional derivatives are considered to be the best tool for characterizing memory and hereditary properties of different materials and processes. Thus, the benefits of fractional derivatives are far more rather than classical integer-order models.

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