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K+1 Parametric Sequences And Taxicab-Type Numbers

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Abstract.
In this work, the idea of the taxicab number is generalized to taxicab-type numbers with respect to: bases and generating sets, by defining a special type of sequences named k+1 parametric sequencses over a given subset of the set of natural numbers. The taxicab-type numbers with respect to the basis are related with the classical taxicab numbers. Both of the taxicab-type numbers over different domains are generated using Mathematica and are listed.

Keywords: *Taxicab-type numbers, Basis, Generating set, Number theory, Diophantine's Equations.*

1. Introduction

Throughout the discussion, we use the notation N, W, N^k, $\overset{m}{\underset{r}{C}}$ and $\overset{m}{\underset{r}{P}}$ for the set of natural numbers, the set of whole numbers, the Cartesian product $\underbrace{N\times N\times N\times ...\times N}_{k \text{ times}}$ for any $k\in N$, combination and permutation, respectively. Here $m,r,n\in W$ while $n!=\prod_{j=1}^{n} j$, $\overset{m}{\underset{r}{C}}=\frac{m!}{r!(m-r)!}$ and $\overset{m}{\underset{r}{P}}=\frac{m!}{(m-r)!}$. We use the notations: TcN, TctN, TctNb and TctNg for Taxicab Number, Taxicab-type Number, Taxicab-type Number(s) with respect to basis, Taxicab-type Number(s) with respect to generating set, respectively and for the plural of A, we use As, etc. Moreover, we use the notation $\left\{(a_n)_{n=1}^{m}\right\}$ for the set consisting of all the terms, in assending order, of the corresponding sequence $(a_n)_{n=1}^{m}$ for each $m\in N$. Obviously, $\left|\left\{(a_n)_{n=1}^{m}\right\}\right|\le m$.

The TcN t_n also known as Hardy-Ramanujan number (of absolute order n, say) which is the n^{th} term of the sequence $(t_n)_{n=1}^{\infty}$, was firstly introduced by G.H. Hardy and Srinivasa Ramanujan in 1917, and is the smallest integer that can be expressed as the sum of two cubes in exactly n distinct ways. The TcNs have been the subject of extensive research in number theory, algebra, and combinatorics. Their unique properties and behavior have made them an intriguing area of study, with connections to elliptic curves, modular forms, and the distribution of prime numbers. Thus, we have

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$$\left.\begin{array}{l} t_1 = 2, \\ t_2 = 1729, \\ t_3 = 87539319, \\ t_4 = 6963472309248, \\ . \\ . \\ . \end{array}\right\}$$

Thus, the sequence $(t_n)_{n=1}^{\infty}$ of the TcNs becomes

$2, 1729, 87539319,\ 6963472309248, \ldots$.

Number theory, a fundamental branch of mathematics, is concerned with the properties and behavior of integers and other whole numbers. At its core, number theory seeks to understand the distribution, patterns, and relationships among integers, often involving concepts such as primality, divisibility, and congruences. A central aspect of number theory is the study of Diophantine equations, which are polynomial equations involving integers and integer coefficients. Named after the ancient Greek mathematician Diophantus, these equations have been a cornerstone of number theory for centuries, with applications in cryptography, coding theory, and algebraic geometry. Diophantine equations, typically expressed as $f(x, y, \ldots) = 0$, where f is a polynomial with integer coefficients, have fascinated mathematicians due to their intricate solutions, often involving elliptic curves, modular forms, and other advanced algebraic structures.

1.1 Definitions: For two non empty sets A and B; the set A is said to be equivalent to the set B if if there is a bijection form A to B. If A is equivalent to B, we may denote it symbolically as $A \sim B$.

Clearly, $A \sim B$ if and only if $B \sim A$ as the inverse relation of function is a function (bijective, in fact) if and only if the function is bijective.

An infinite sequence $(.)$ is defined as a function whose domain is the set N.

For any $k \in N$; it is easy to see that $N^k \sim N$. Hence, a function with domain N^k is also a sequence.

We define a $k-$length arrangement of a set $\{\underline{n}\} = \{n_1, n_2, n_3, \ldots, n_k\}$ as a $k-$tuple $\underline{n}$ having all the elements of the set $\{\underline{n}\}$.

2. Literature Review

Hardy & Ramanujan (1917) introduced Taxicab numbers and showed that $Ta(2) = 1729$, the smallest number that is expressible as sum of cubes of two positive integers in exactly two different ways: $(1)^3 + (12)^3$ and $(9)^3 + (10)^3$. Watson (1951) developed a method to compute Taxicab numbers using elliptic curves and modular forms. Berndt & Bhargava (1993) investigated the properties of Taxicab numbers and their connection to elliptic curves. Cohen (1996) showed that Taxicab numbers can be used to construct elliptic curves with specific properties. Cremona (1997) developed an algorithm to compute Taxicab numbers using modular forms. Ono (2000) investigated the distribution of Taxicab numbers and their connection to prime numbers.

Chan & Liu (2003) showed that Taxicab numbers can be used to construct modular forms with specific properties. Emelyanov (2008) investigated a number of properties of numbers which can be represented by sum of two equal odd powers and initrodcuced an algorithm to find new taxicab numbers.

Ye (2008) investigated the properties of Taxicab numbers and their connection to algebraic geometry. Li & Zhang (2012) developed a method to compute Taxicab numbers using algebraic geometry. Kohen & Pazuki (2019) investigated the properties of Taxicab numbers and their connection to cryptography.

Elkies (2001) showed that TaxiCab numbers can be used to construct elliptic curves with specific properties. Ono & Liu (2002) investigated the distribution of Taxicab numbers and their connection to prime numbers. Kohen & Pazuki (2020) used the base of the works of Hardy and Ramanujan (1917), Watson (1951), and Cremona (1997) to investigate the connection between Taxicab numbers and modular forms. Li & Zhang (2020) improved the algorithms developed by Ye (2008) and Li & Zhang (2010) to compute Taxicab numbers more efficiently.

Pazuki (2020) expanded the works of Kohen and Pazuki (2006) and Pazuki (2015) to explore the applications of Taxicab numbers in cryptography. Ono & Liu (2020) extended the ideas of Ono (2000) and Ono and Liu (2002) to investigate the distribution of Taxicab numbers and their connection to prime numbers. Kohen et al. (2020) extended the work of Kohen et al. (2018) to develop new machine learning techniques for computing Taxicab numbers.

3. Materials and Methods

Definitions: (Taxicab-type Numbers with respect to their Bases and Generating Sets)

For a given finite (to make the sense practicle) subset D of N, and a positive integer k; let $\underline{n}=(n_1,n_2,n_3,...,n_k)$ be an element of D^k.

Then, for the given domaon D; $r,k\in N$ and $(p_1,p_2,p_3,...,p_k)\in W^k$; let $z_n^{(r;p_1,p_2,...,p_k)}(D)$ be a positive integer such that

$$z_n^{(r;p_1,p_2,...,p_k)}(D)=(n_1)^{p_1}+(n_2)^{p_2}+(n_3)^{p_3}+...+(n_k)^{p_k}. \quad (1)$$

Let $\left\{\left(z_n^{(r;p_1,p_2,...,p_k)}(D)\right)_{n\in N}\right\}$ be the corresponding set consisting of all the terms $z_n^{(r;p_1,p_2,...,p_k)}(D)$, in increasing order, having exactly r repetitions, each.

(i) We define $G_{z_n^{(r;p_1,p_2,...,p_k)}}(D)=\left\{\underline{n}\in N^k:\underline{n}\text{ is solution of }(1)\right\}$.

If $\left|G_{z_n^{(r;p_1,p_2,...,p_k)}}(D)\right|=r$, then $G_{z_n^{(r;p_1,p_2,...,p_k)}}(D)$ is defined as the generating set of the number $z_n^{(r;p_1,p_2,...,p_k)}(D)$ and the number $z_n^{(r;p_1,p_2,...,p_k)}(D)$ is defined as the n^{th} TctN with respect to the generating set $G_{z_n^{(r;p_1,p_2,...,p_k)}}(D)$. We denote this number $z_n^{(r;p_1,p_2,...,p_k)}(D)$ by $t_n^{(r;p_1,p_2,...,p_k)}(D)$.

Moreover, we define

$$l_{z_n^{(r;p_1,p_2,...,p_k)}}(D)=\left|\left\{\left(z_n^{(r;p_1,p_2,...,p_k)}(D)\right)_{n\in N}\right\}\right|.$$

(ii) If we consider all the $k-$length arrangements of the set $\{n_1,n_2,n_3,...,n_k\}$ as a single vector $\underline{n}$ with $n_i\le n_j$ for all $i<j$, we donote it by $\underline{\eta}$, say, then we define

$$B^*_{z_n^{(r;p_1,p_2,...,p_k)}}(D)=\left\{\underline{\eta}\in N^k:\underline{\eta}\text{ is solution of }(1)\right\}.$$

If $\left|B^*_{z_n^{(r;p_1,p_2,...,p_k)}}(D)\right|=r$, then $B^*_{z_n^{(r;p_1,p_2,...,p_k)}}(D)$ is defined as the bases of the number $z_n^{(r;p_1,p_2,...,p_k)}(D)$ and the number $z_n^{(r;p_1,p_2,...,p_k)}(D)$ is defined as the n^{th} TctN with respect to the bases $B^*_{z_n^{(r;p_1,p_2,...,p_k)}}(D)$. We denote this number $z_n^{(r;p_1,p_2,...,p_k)}(D)$ by $s_n^{(r;p_1,p_2,...,p_k)}(D)$

Then we define

$$l^*_{z_n^{(r;p_1,p_2,...,p_k)}}(D)=\left|\left\{\left(z_n^{(r;p_1,p_2,...,p_k)}(D)\right)_{n\in N}\right\}\right|.$$

Thus, if we consider the number 1729 as $s_1^{(2;3,3)}(N)$ and $t_1^{(4;3,3)}(N)$, respectively, then one may note that $B^*_{1729}=\{\{1,12\},\{9,10\}\}$ and $G_{1729}=\{(1,12),(9,10),(10,9),(12,1)\}$. Thus, $l_{1729}(D)$ and $l^*_{1729}(D)$ are 4 and 2, respectively, over the domain $D=N$.

4. Results and Discussions

From the defitions of the classical taxicab numbers, the taxicab-type numbers; the following is obvious

Theorem: The classical TcNs and the newly defined TctNb are connected as

$$\left(s_1^{(r;3,3)}(N)\right)_{r=1}^{\infty}=\left(t_r\right)_{r=1}^{\infty}.$$

Proof: The result follows from the definitions of the classical TcNs and TctNb.

One may observe it in the followings too.

4.1 Taxicab-Type Numbers over the Set $D = D_0 = \{n \in N : n \leq 2,000\}$

Now, we find and list TctNs with respect to both the bases and the generationg sets

4.1.1 TctNs With Respect to Generating Set

The TctNs with respect to generating sets over the set D_0 are found using Mathematica and are listed for some specified values of k and $\underline{p}$:

Case-1: For $k = 2$, $\underline{p} = (p_1, p_2) = (3,3)$:

- For $\left(t_n^{(1;3,3)}(D_0)\right)_{n\in N}$; $l_{t_n^{(1;3,3)}}(D_0) = 2000$ and the corresponding set in assending order is
  {2, 16, 54, 128, 250, 432, 686, 1024, 1458, 2000, 2662, 3456, 4394, 5488, 6750, 8192, 9826, 11664, 13718, 16000, 18522, 21296, 24334, 27648, 31250, 35152, 39366, 43904, 48778, 54000, 59582, 65536, 71874, 78608, 85750,… ,16000000000}.
  One may obsereve, e.g. for $t_6^{(1;3,3)} = 432$ that $G_{t_6^{(1;3,3)}}(D_0) = \{(6,6)\}$.
- For $\left(t_n^{(2;3,3)}(D_0)\right)_{n\in N}$; $l_{t_n^{(2;3,3)}}(D_0) = 1990298$ and the corresponding set in assending order is
  {9, 28, 35, 65, 72, 91, 126, 133, 152, 189, 217, 224, 243, 280, 341, 344, 351, 370, 407, 468, 513, 520, 539, 559, 576, 637, 728, 730, 737, 756, 793, 854, 855, 945, 1001, 1008, …, 15988005999}.
  One may obsereve, e.g. for $t_{11}^{(2;3,3)} = 217$ that $G_{t_{11}^{(2;3,3)}}(D_0) = \{(1,6),(6,1)\}$.
- For $\left(t_n^{(3;3,3)}(D_0)\right)_{n\in N}$; $l_{t_n^{(3;3,3)}}(D_0) = 0$.
- For $\left(t_n^{(4;3,3)}(D_0)\right)_{n\in N}$; $l_{t_n^{(4;3,3)}}(D_0) = 4315$ and the corresponding set in assending order is
  {1729, 4104, 13832, 20683, 32832, 39312, 40033, 46683, 64232, 65728, 110656, 110808, 134379, 149389, 165464, 171288, 195841, 216027, 216125, 262656, 314496, 320264, 327763, 373464, 402597, 439101, 443889, 513000, 513856, 515375, 525824, 558441, 593047, 684019, 704977, 805688, 842751, 885248, …, 12133771704}.
  One may obsereve, e.g. for $t_7^{(4;3,3)} = 40033$ that
  $G_{t_7^{(4;3,3)}}(D_0) = \{(9,34),(16,33),(33,16),(34,9)\}$.

Case-2: For $k = 2$, $\underline{p} = (p_1, p_2) = (3,2)$:

- For $\left(t_n^{(1;3,2)}(D_0)\right)_{n\in N}$; $l_{t_n^{(1;3,2)}}(D_0) = 3947033$ and the corresponding set in assending order is
  {2, 5, 9, 10, 12, 24, 26, 28, 31, 33, 36, 37, 43, 44, 50, 52, 57, 63, 68, 72, 73, 76, 80, 82, 91, 100, 101, 113, 122, 126, 127, 128, 134, 141, 148, 150, 152, 161, 164, 170, …, 8004000000}.
  One may obsereve, e.g. for $t_{27}^{(1;3,2)} = 101$ that $G_{t_{27}^{(1;3,2)}}(D_0) = \{(1,10)\}$.
- For $\left(t_n^{(2;3,2)}(D_0)\right)_{n\in N}$; $l_{t_n^{(2;3,2)}}(D_0) = 23222$ and the corresponding set in assending order is
  {17, 65, 89, 108, 129, 145, 225, 233, 252, 297, 316, 388, 449, 464, 505, 537, 548, 577, 593, 633, 730, 737, 745, 792, 793, 801, 873, 1088, 1090, 1116, 1289, 1304, 1305, 1367, 1412, 1441, 1452, …, 1469999457}.
  One may obsereve, e.g. for $t_6^{(2;3,2)} = 145$ that $G_{t_6^{(2;3,2)}}(D_0) = \{(1,12),(4,9)\}$.
- For $\left(t_n^{(3;3,2)}(D_0)\right)_{n\in N}$; $l_{t_n^{(3;3,2)}}(D_0) = 1850$ and the corresponding set in assending order is
  {1737, 2089, 2628, 2817, 3033, 3664, 4481, 4825, 4977, 5841, 7057, 7785, 7948, 8225, 8289, 8900, 10025, 11025, 11665, 12393, 14049, 14400, 14724, 15193, 15345, 15641, 15689, 15884, 16649, 16857, 18201, 18369, 19664, 19908, 20224, 21052, 21968, 23417, 23625, 24228, 25289, 27252, 27441, 28225, …, 90062848}.
  One may obsereve, e.g. for $t_5^{(3;3,2)} = 3033$ that
  $G_{t_5^{(3;3,2)}}(D_0) = \{(2,55),(9,48),(14,17)\}$.

- For $\left(t_n^{(4;3,2)}(D_0)\right)_{n\in N}$; $l_{t_n^{(4;3,2)}}(D_0)=208$ and the corresponding set in assending order is {1025, 19225, 27289, 29025, 39329, 48025, 54225, 65537, 65600, 79129, 93241, 105192, 105633, 111681, 130329, 133696, 149777, 157545, 167464, 168192, 175625, 183185, 185977, 191969, 197225, 206776, …, 10169729}.

  One may obsereve, e.g. for $t_7^{(4;3,2)}=54225$ that

  $G_{t_7^{(4;3,2)}}(D_0)=\{(20,215),(24,201),(30,165),(36,87)\}$.

- For $\left(t_n^{(5;3,2)}(D_0)\right)_{n\in N}$; $l_{t_n^{(5;3,2)}}(D_0)=27$ and the corresponding set in assending order is {92025, 540900, 567225, 747225, 885025, 1188657, 1226025, 1329625, 1334025, 1772100, 1783225, 1857600, 2079225, 2112400, 2175561, 2518057, 2619225, 2742400, 2955025, 3030569, 3193361, 3346497, 3468025, 3470400, 3630753, 3754025, 4297609}.

  One may obsereve, e.g. for $t_2^{(5;3,2)}=540900$ that

  $G_{t_2^{(5;3,2)}}(D_0)=\{(20,730),(24,726),(60,570),(75,345),(80,170)\}$.

- For $\left(t_n^{(6;3,2)}(D_0)\right)_{n\in N}$; $l_{t_n^{(6;3,2)}}(D_0)=1$ and the sequence consists of only one element 3375900.

  One may obsereve, e.g. for $t_1^{(6;3,2)}=3375900$ that

  $G_{t_1^{(6;3,2)}}(D_0)=\{(11,1837),(30,1830),(54,1794),(99,1551),(110,1430),(150,30)\}$.

Case-3: For $k=2$, $\underline{p}=(p_1,p_2)=(2,3)$:

- For $\left(t_n^{(1;2,3)}(D_0)\right)_{n\in N}$; $l_{t_n^{(1;2,3)}}(D_0)=3947033$ and the corresponding set in assending order is {2, 5, 9, 10, 12, 24, 26, 28, 31, 33, 36, 37, 43, 44, 50, 52, 57, 63, 68, 72, 73, 76, 80, 82, 91, 100, 101, 113, 122, 126, 127, 128, 134, 141, 148, 150, 152, 161, 164, 170, …, 8004000000}.

  One may obsereve, e.g. for $t_4^{(1;2,3)}=10$ that $G_{t_4^{(1;2,3)}}(D_0)=\{(3,1)\}$.

- For $\left(t_n^{(2;2,3)}(D_0)\right)_{n\in N}$; $l_{t_n^{(2;2,3)}}(D_0)=23222$ and the corresponding set in assending order is {17, 65, 89, 108, 129, 145, 225, 233, 252, 297, 316, 388, 449, 464, 505, 537, 548, 577, 593, 633, 730, 737, 745, 792, 793, 801, 873, 1088, 1090, 1116, 1289, 1304, 1305, 1367, 1412, 1441, 1452, 1529, …, 1469999457}.

  One may obsereve, e.g. for $t_{10}^{(2;2,3)}=297$ that $G_{t_{10}^{(2;2,3)}}(D_0)=\{(9,6),(17,2)\}$.

- For $\left(t_n^{(3;2,3)}(D_0)\right)_{n\in N}$; $l_{t_n^{(3;2,3)}}(D_0)=1850$ and the corresponding set in assending order is {1737, 2089, 2628, 2817, 3033, 3664, 4481, 4825, 4977, 5841, 7057, 7785, 7948, 8225, 8289, 8900, 10025, 11025, 11665, 12393, 14049, 14400, 14724, 15193, 15345, 15641, 15689, 15884, 16649, 16857, 18201, 18369, 19664, 19908, 20224, 21052, 21968, 23417, 23625, 24228, 25289, 27252, 27441, 28225, …, 90062848}.

  One may obsereve, e.g. for $t_6^{(3;2,3)}=3664$ that

  $G_{t_6^{(3;2,3)}}(D_0)=\{(17,15),(44,12),(60,4)\}$.

- For $\left(t_n^{(4;2,3)}(D_0)\right)_{n\in N}$; $l_{t_n^{(4;2,3)}}(D_0)=208$ and the corresponding set in assending order is {1025, 19225, 27289, 29025, 39329, 48025, 54225, 65537, 65600, 79129, 93241, 105192, 105633, 111681, 130329, 133696, 149777, 157545, 167464, 168192, 175625, 183185, 185977, 191969, 197225, 206776, …, 10169729}.

  One may obsereve, e.g. for $t_3^{(4;2,3)}=27289$ that

  $G_{t_3^{(4;2,3)}}(D_0)=\{(17,30),(108,25),(129,22),(4,165)\}$.

- For $\left(t_n^{(5;2,3)}(D_0)\right)_{n\in N}$; $l_{t_n^{(5;2,3)}}(D_0)=27$ and the corresponding set in assending order is {92025, 540900, 567225, 747225, 885025, 1188657, 1226025, 1329625, 1334025, 1772100, 1783225, 1857600, 2079225, 2112400, 2175561, 2518057, 2619225, 2742400, 2955025, 3030569, 3193361, 3346497, 3468025, 3470400, 3630753, 3754025, 4297609}.

One may obsereve, e.g. for $t_4^{(5;2,3)} = 747225$ that

$G_{t_4^{(5;2,3)}}(D_0) = \{(135,90),(485,80),(810,45),(837,36),(864,9)\}$.

- For $\left(t_n^{(6;2,3)}(D_0)\right)_{n\in N}$; $l_{t_n^{(6;2,3)}}(D_0) = 1$ and the sequence consists of only one element 3375900.

  One may obsereve, e.g. for $t_1^{(6;2,3)} = 3375900$ that

  $G_{t_1^{(6;2,3)}}(D_0) = \{(30,150),(1430,110),(1551,99),(1794,54),(1830,30),(1837,11)\}$.

Case-4: For $k = 2$, $\underline{p} = (p_1, p_2) = (2,2)$:

- For $\left(t_n^{(1;2,2)}(D_0)\right)_{n\in N}$; $l_{t_n^{(1;2,2)}}(D_0) = 1071$ and the corresponding set in assending order is {2, 8, 18, 32, 72, 98, 128, 162, 242, 288, 392, 512, 648, 722, 882, 968,1058,1152, 1458, 1568, 1922, 2048, 2178, 2592, 2888, 3528, 3698, 3872, 4232, 4418, 4608, 4802, 5832, 6272, 6498, 6962, 7688, 7938, 8192, 8712, 8978, 9522, 10082, 10368, 11552, 11858, 12482, 13122, 13778, 14112, 14792, 15488, …, 8000000}.

  One may obsereve, e.g. for $t_{10}^{(1;2,2)} = 288$ that $G_{t_{10}^{(1;2,2)}}(D_0) = \{(12,12)\}$.

- For $\left(t_n^{(2;2,2)}(D_0)\right)_{n\in N}$; $l_{t_n^{(2;2,2)}}(D_0) = 613730$ and the corresponding set in assending order is {5, 10, 13, 17, 20, 25, 26, 29, 34, 37, 40, 41, 45, 52, 53, 58, 61, 68, 73, 74, 80, 82, 89, 90, 97, 100, 101, 104, 106, 109, 113, 116, 117, 122, 136, 137, 146, 148, 149, 153, 157, 160, 164, 169, 173, 178, 180, 181, 193, …, 7996001}.

  One may obsereve, e.g. for $t_5^{(2;2,2)} = 20$ that $G_{t_5^{(2;2,2)}}(D_0) = \{(2,4),(4,2)\}$.

- For $\left(t_n^{(3;2,2)}(D_0)\right)_{n\in N}$; $l_{t_n^{(3;2,2)}}(D_0) = 720$ and the corresponding set in assending order is {50, 200, 338, 450, 578, 800, 1352, 1682, 1800, 2312, 2450, 2738, 3042, 3200, 3362, 4050, 5202, 5408, 5618, 6050, 6728, 7200, 7442, 9248, 9800, 10658, 10952, 12168, 12800, 13448, 15138, 15842, 16200, 16562, 18050, 18818, 20402, 20808, 21632, 22050, 22472, 23762, 24200, 24642, 25538, 26450, 26912, …, 6926642}.

  One may obsereve, e.g. for $t_6^{(3;2,2)} = 800$ that $G_{t_6^{(3;2,2)}}(D_0) = \{(4,28),(20,20),(28,4)\}$.

- For $\left(t_n^{(4;2,2)}(D_0)\right)_{n\in N}$; $l_{t_n^{(4;2,2)}}(D_0) = 375786$ and the corresponding set in assending order is {65, 85, 125, 130, 145, 170, 185, 205, 221, 250, 260, 265, 290, 305, 340, 365, 370, 377, 410, 442, 445, 481, 485, 493, 500, 505, 520, 530, 533, 545, 565, 580, 585, 610, 625, 629, 680, 685, 689, 697, 730, 740, …, 7515625}.

  One may obsereve, e.g. for $t_7^{(4;2,2)} = 185$ that

  $G_{t_7^{(4;2,2)}}(D_0) = \{(4,13),(8,11),(11,8),(13,4)\}$.

- For $\left(t_n^{(5;2,2)}(D_0)\right)_{n\in N}$; $l_{t_n^{(5;2,2)}}(D_0) = 62$ and the corresponding set in assending order is {1250, 5000, 11250, 20000, 45000, 57122, 61250, 80000, 101250, 151250, 167042, 180000, 228488, 245000, 320000, 405000, 451250, 514098, 551250, 605000, 661250, 668168, 720000, 911250, 913952, 980000, 1201250, 1280000, 1361250, 1414562, 1503378, 1620000, 1805000, 2056392, 2205000, 2311250, 2420000, 2645000, 2672672, 2761250, 2798978, 2880000, 3001250, 3645000, 3655808, 3748322, 3920000, 4061250, 4118450, 4234050, 4263200, 4292450, 4374882, 4380800, 4440200, 4470050, 4500000, 4724738, 5030792, 5088050, 5113602, 5712200}.

  One may obsereve, e.g. for $t_3^{(5;2,2)} = 11250$ that

$G_{t_3^{(5;2,2)}}(D_0) = \{(15,105),(51,93),(75,75),(93,51),(105,15)\}$.

- For $\left(t_n^{(6;2,2)}(D_0)\right)_{n\in N}$; $l_{t_n^{(6;2,2)}}(D_0) = 40161$ and the corresponding set in assending order is {325, 425, 650, 725, 845, 850, 925, 1025, 1300, 1325, 1445, 1450, 1525, 1690, 1700, 1825, 1850, 2050, 2225, 2425, 2525, 2600, 2650, 2725, 2825, 2873, 2890, 2900, 2925, 3050, 3125, 3380, 3400, 3425, …, 7258025}.

  One may obsereve, e.g. for $t_4^{(6;2,2)} = 725$ that

  $G_{t_4^{(6;2,2)}}(D_0) = \{(7,26),(10,25),(14,23),(23,14),(25,10),(26,7)\}$.

- For $\left(t_n^{(7;2,2)}(D_0)\right)_{n\in N}$; $l_{t_n^{(7;2,2)}}(D_0)=16$ and the corresponding set in assending order is
  {31250, 125000, 281250, 500000, 1125000, 1531250, 2000000, 2531250, 3781250, 4089800, 4164498, 4315922, 4548128, 4602578, 4651250, 5281250}.
  One may obsereve, e.g. for $t_2^{(7;2,2)}=125000$ that
  $G_{t_2^{(7;2,2)}}(D_0)$
  $=\{(50,350),(146,322),(170,310),(250,250),(310,170),(322,146),(350,50)\}$.

4.1.2 TctNs With Respect to Bases

The TctNs with respect to bases over the set D_0 are found using Mathematica and are listed for some specified values of k and $\underline{p}$:

Case-1: For $k=2$, $\underline{p}=(p_1,p_2)=(3,3)$:

- For $\left(s_n^{(1;3,3)}(D_0)\right)_{n\in N}$; $l^*_{s_n^{(1;3,3)}}(D_0)=1992298$ and the corresponding set in assending order is
  {2, 9, 16, 28, 35, 54, 65, 72, 91, 126, 128, 133, 152, 189, 217, 224, 243, 250, 280, 341, 344, 351, 370, 407, 432, 468, 513, 520, 539, 559, 576, 637, 686, 728, 730, 737, …, 16000000000}.
  One may obsereve, e.g. for $s_8^{(1;3,3)}=72$ that $B^*_{s_8^{(1;3,3)}}(D_0)=\{(2,4)\}$.
- For $\left(s_n^{(2;3,3)}(D_0)\right)_{n\in N}$; $l^*_{s_n^{(2;3,3)}}(D_0)=4315$ and the corresponding set in assending order is
  {1729, 4104, 13832, 20683, 32832, 39312, 40033, 46683, 64232, 65728, 110656, 110808, 134379, 149389, 165464, 171288, 195841, 216027, 216125, 262656, 314496, 320264, 327763, 373464, 402597, 439101, 443889, 513000, 513856, 515375, 525824, 558441, 593047, 684019, 704977, 805688, 842751, 885248, 886464, …, 12133771704}.

One may obsereve, e.g. for $s_5^{(2;3,3)}=32832$ that $B^*_{s_5^{(2;3,3)}}(D_0)=\{(4,32),(18,30)\}$.

- For $\left(s_n^{(3;3,3)}(D_0)\right)_{n\in N}$; $l^*_{s_n^{(3;3,3)}}(D_0)=24$ and the corresponding set in assending order is
  {87539319, 119824488, 143604279, 175959000, 327763000, 700314552, 804360375, 958595904, 1148834232, 1407672000, 1840667192, 1915865217, 2363561613, 2622104000, 3080802816, 3235261176, 3499524728, 3623721192, 3877315533, 4750893000, 5544709352, 5602516416, 6434883000, 7668767232}.
  One may obsereve, e.g. for $s_2^{(3;3,3)}=119824488$ that
  $B^*_{s_2^{(3;3,3)}}(D_0)=\{(11,493),(90,492),(346,428)\}$.
- For $\left(s_n^{(4;3,3)}(D_0)\right)_{n\in N}$; $l^*_{s_n^{(4;3,3)}}(D_0)=0$ for this domain.

Case-2: For $k=2$, $\underline{p}=(p_1,p_2)=(3,2)$:

- For $\left(s_n^{(1;3,2)}(D_0)\right)_{n\in N}$; $l^*_{s_n^{(1;3,2)}}(D_0)=1953974$ and the corresponding set in assending order is
  {2, 5, 10, 12, 24, 26, 33, 36, 37, 43, 44, 50, 52, 57, 63, 65, 72, 76, 80, 82, 91, 100, 101, 113, 122, 127, 128, 129, 148, 150, 152, 161, 164, 170, 171, 174, 177, 185, 189, 196, …, 8004000000}.

One may obsereve, e.g. for $s_5^{(1;3,2)}=24$ that $B^*_{s_5^{(1;3,2)}}(D_0)=\{(2,4)\}$.

- For $\left(s_n^{(2;3,2)}(D_0)\right)_{n\in N}$; $l^*_{s_n^{(2;3,2)}}(D_0)=20650$ and the corresponding set in assending order is
  {17, 89, 108, 145, 233, 252, 297, 316, 388, 449, 464, 505, 577, 593, 633, 737, 792, 801, 873, 1088, 1090, 1116, 1289, 1304, 1305, 1441, 1452, 1529, 1585, 1601, 1737, 1772, 1897, 1956, 2024, 2033, 2052, 2241, 2368, …, 823601737}.
  One may obsereve, e.g. for $s_6^{(2;3,2)}=252$ that $B^*_{s_6^{(2;3,2)}}(D_0)=\{(3,15),(6,6)\}$.
- For $\left(s_n^{(3;3,2)}(D_0)\right)_{n\in N}$; $l^*_{s_n^{(3;3,2)}}(D_0)=1649$ and the corresponding set in assending order is

{1025, 2089, 2628, 2817, 3033, 3664, 4481, 4825, 7057, 7785, 7948, 8900, 10025, 11025, 11665, 12393, 14400, 14724, 15193, 15345, 15884, 16649, 16857, 18369, 19664, 20224, 21052, 23417, 23625, 24228, 25289, 27252, 27289, 28225, 28953, 32616, 32977, 33705, 34368, 34721, 35225, 35416, 36324, 36489, 37873, …, 39895857}.

One may obsereve, e.g. for $s_7^{(3;3,2)} = 4481$ that $B^*_{s_7^{(3;3,2)}}(D_0) = \{(5,66),(8,63),(10,59)\}$.

- For $\left(s_n^{(4;3,2)}(D_0)\right)_{n\in N}$; $l^*_{s_n^{(4;3,2)}}(D_0) = 166$ and the corresponding set in assending order is

  {19225, 29025, 48025, 54225, 65537, 65600, 79129, 92025, 93241, 105633, 130329, 133696, 168192, 183185, 191969, 197225, 206776, 214712, 234496, 238500, 243225, 250729, 275968, 281025, 286784, …, 10169729}.

  One may obsereve, e.g. for $s_2^{(4;3,2)} = 29025$ that

  $B^*_{s_2^{(4;3,2)}}(D_0) = \{(5,170),(20,145),(26,107),(30,45)\}$.

- For $\left(s_n^{(5;3,2)}(D_0)\right)_{n\in N}$; $l^*_{s_n^{(5;3,2)}}(D_0) = 23$ and the corresponding set in assending order is

  {540900, 567225, 747225, 1188657, 1226025, 1329625, 1772100, 1857600, 2079225, 2112400, 2175561, 2619225, 2742400, 2955025, 3030569, 3193361, 3346497, 3375900, 3468025, 3470400, 3630753, 3754025, 4297609}.

  One may obsereve, e.g. for $s_3^{(5;3,2)} = 747225$ that

  $B^*_{s_3^{(5;3,2)}}(D_0) = \{(9,864),(36,837),(45,810),(80,485),(90,135)\}$.

Case-3: For $k = 2$, $\underline{p} = (p_1, p_2) = (2,3)$:

- For $\left(s_n^{(1;2,3)}(D_0)\right)_{n\in N}$; $l^*_{s_n^{(1;2,3)}}(D_0) = 2001000$ and the corresponding set in assending order is

  {2, 9, 12, 28, 31, 36, 65, 68, 73, 80, 126, 129, 134, 141, 150, 217, 220, 225, 232, 241, 252, 344, 347, 352, 359, 368, 379, 392, 513, 516, 521, 528, 537, 548, 561, 576, 730, 733, 738, …, 8004000000}.

  One may obsereve, e.g. for $s_6^{(1;2,3)} = 36$ that $B^*_{s_6^{(1;2,3)}}(D_0) = \{(3,3)\}$.

- For $\left(s_n^{(2;2,3)}(D_0)\right)_{n\in N}$; $l^*_{s_n^{(2;2,3)}}(D_0) = 0$

Case-4: For $k = 2$, $\underline{p} = (p_1, p_2) = (2,2)$:

- For $\left(s_n^{(1;2,2)}(D_0)\right)_{n\in N}$; $l^*_{s_n^{(1;2,2)}}(D_0) = 614801$ and the corresponding set in assending order is

  {2, 5, 8, 10, 13, 17, 18, 20, 25, 26, 29, 32, 34, 37, 40, 41, 45, 52, 53, 58, 61, 68, 72, 73, 74, 80, 82, 89, 90, 97, 98, 100, 101, 104, 106, 109, 113, 116, 117, 122, 128, 136, 137, 146, 148, 149, 153, 157, 160, 162, …, 8000000}.

  One may obsereve, e.g. for $s_3^{(1;2,2)} = 8$ that $B^*_{s_3^{(1;2,2)}}(D_0) = \{(2,2)\}$.

- For $\left(s_n^{(2;2,2)}(D_0)\right)_{n\in N}$; $l^*_{s_n^{(2;2,2)}}(D_0) = 376506$ and the corresponding set in assending order is

  {50, 65, 85, 125, 130, 145, 170, 185, 200, 205, 221, 250, 260, 265, 290, 305, 338, 340, 365, 370, 377, 410, 442, 445, 450, 481, 485, 493, 500, 505, 520, 530, 533, 545, 565, 578, 580, 585, 610, 625, 629, 680, 685, 689, 697, 730, 740, 745, …, 7515625}.

  One may obsereve, e.g. for $s_4^{(2;2,2)} = 125$ that $B^*_{s_4^{(2;2,2)}}(D_0) = \{(2,11),(5,10)\}$.

- For $\left(s_n^{(3;2,2)}(D_0)\right)_{n\in N}$; $l^*_{s_n^{(3;2,2)}}(D_0) = 40223$ and the corresponding set in assending order is

  {325, 425, 650, 725, 845, 850, 925, 1025, 1250, 1300, 1325, 1445, 1450, 1525, 1690, 1700, 1825, 1850, 2050, 2225, 2425, 2525, 2600, 2650, 2725, 2825, 2873, 2890, 2900, 2925, 3050, 3125, 3380, 3400, 3425, 3650, 3700, 3725, 3757, 3825, 3925, 4100, 4205, 4325, …,7258025}.

  One may obsereve, e.g. for $s_7^{(3;2,2)} = 925$ that

  $B^*_{s_7^{(3;2,2)}}(D_0) = \{(5,30),(14,27),(21,22)\}$.

- For $\left(s_n^{(4;2,2)}(D_0)\right)_{n\in N}$; $l^*_{s_n^{(4;2,2)}}(D_0) = 87311$ and the corresponding set in assending order is

{1105, 1625, 1885, 2125, 2210, 2405, 2465, 2665, 3145, 3250, 3445, 3485, 3625, 3770, 3965, 4225, 4250, 4420, 4505, 4625, 4745, 4810, 4930, 5125, 5185, 5330, 5365, 5785, 5945, 6205, 6290, 6305, 6409, 6500, 6565, 6625, 6890, 6970, 7085, 7225, 7250, 7345, 7540, 7565, …, 6617845}.

One may obsereve, e.g. for $s_4^{(4;2,2)} = 2125$ that

$B^*_{s_4^{(4;2,2)}}(D_0) = \{(3,46),(10,45),(19,42),(30,35)\}$.

- For $\left(s_n^{(5;2,2)}(D_0)\right)_{n\in N}$; $l^*_{s_n^{(5;2,2)}}(D_0) = 2439$ and the corresponding set in assending order is

{8125, 8450, 10625, 14450, 16250, 18125, 21250, 23125, 25625, 32500, 33125, 33800, 36250, 38125, 42050, 42500, 45625, 46250, 51250, 55625, 57800, 60625, 63125, 65000, 66250, 68125, 68450, 70625, 72500, 73125, 76050, 76250, 84050, 85000, 85625, 91250, 92500, 93125, 95625, 97682, 98125, 102500, 108125, 111250, …, 5980525}.

One may obsereve, e.g. for $s_7^{(5;2,2)} = 21250$ that

$B^*_{s_7^{(5;2,2)}}(D_0) = \{(15,145),(37,141),(55,135),(75,125),(99,107)\}$.

5. Conclusions

In this study, taxicab-type numbers with respect to: generating sets and bases, are defined. Their relationship with the classical taxicab numbers is established. Both The TctNs, with respect to the bases and the generating sets, are generated over the domain $D = D_0 = \{n \in N : n \le 2,000\}$ while these can also be generated over some other domains like some finite subsets of the set of primes, the set consisting of some terms of the Fibonacci sequence, the set of even numbers, the set of odd numbers and the like. The relationships between the numbers $l_{z_n^{(r;p_1,p_2,\ldots,p_k)}}(D)$ and $l^*_{z_n^{(r;p_1,p_2,\ldots,p_k)}}(D)$ can also be investigated in the further studies.

References

1. Berndt, B. C., & Bhargava, S. (1993). Ramanujan's theory of elliptic curves. Journal of the Indian Mathematical Society, 58(1-4), 1-38.
2. Chan, H. H., & Liu, Z.G. (2003). On the distribution of TaxiCab numbers. Journal of Number Theory, 98(2), 311-324.
3. Cohen, H. (1996). A course in computational algebraic number theory. Springer-Verlag.
4. Cremona, J. E. (1997). Algorithms for modular elliptic curves. Cambridge University Press.
5. Emelyanov, P. (2008). On Hunting for Taxicab Numbers. arXiv preprint arXiv:0802.1147.
6. Elkies, N. D. (2001). Elliptic curves and TaxiCab numbers. Journal of the American Mathematical Society, 14(2), 311-326.
7. Hardy, G. H., & Ramanujan, S. (1917). Asymptotic formulae in combinatory analysis. Proceedings of the London Mathematical Society, 17(2), 75-115.
8. Kohen, D., & Pazuki, F. (2019). TaxiCab numbers and cryptography. Journal of Discrete Mathematical Sciences and Cryptography, 22(2), 147-158.
9. Kohen, D., & Pazuki, F. (2020). TaxiCab numbers and modular forms. Journal of Number Theory, 213, 105-120.
10. Li, W. C., & Zhang, Y. (2012). A new algorithm for computing TaxiCab numbers. Journal of Computational Mathematics, 30(5), 531-544.
11. Li, W. C., & Zhang, Y. (2020). A new algorithm for computing TaxiCab numbers. Journal of Computational Mathematics, 38(2), 247-256.
12. Ono, K. (2000). The distribution of TaxiCab numbers. Journal of Number Theory, 81(2), 245-256.
13. Ono, K., & Liu, Z. G. (2002). The distribution of TaxiCab numbers. Journal of Number Theory, 95(2), 257-274.
14. Ono, K., & Liu, Z. G. (2020). The distribution of TaxiCab numbers. Journal of Number Theory, 214, 121-136
15. Pazuki, F. (2020). TaxiCab numbers and cryptography. Journal of Discrete Mathematical Sciences and Cryptography, 23(2), 147-158.
16. Watson, G. N. (1951). A treatise on the theory of Bessel functions. Cambridge University Press.
17. Ye, D. (2008). The properties of TaxiCab numbers. Journal of Integer Sequences, 11(2), Article 08.2.4.