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Exploring Triple-*k* And Penta- *k* Extensions Of Some Generalized Mittag-Leffler Functions And Their Riemann-Liouville *k* -Fractional Integrals

Ghulam Farid[1], Faiza Rehman[2], Aqsa Arage[3], Laraib Naeem[4], Sitara Ajmal[5], Salma Arshad[6]

Abstract.
This study presents a novel $k-$*extension of the generalized classical beta function. Additionally, we introduce a triple*$-k$ *extension and a penta*$-k$ *extension of the generalized classical Mittag-Leffler (ML) function, building upon the extended beta function. These new forms significantly broaden the scope of ML functions. Moreover, we derive the Riemann-Liouville (RL)* $k-$*fractional integrals (FIs) of some functions that incorporate these extended ML functions, further expanding their applications.*

Keywords: *Fractional, integral, derivative, generalized, Riemann-Liouville, Mittag Leffler.*

1. Introduction

The Swedish mathematician Gösta Mittag-Leffler introduced the function in the 1870s while studying the theory of elliptic functions. Mittag-Leffler continued to work on the function, publishing several papers on its properties and applications during the region 1880-1900. From 1900-1920, mathematicians like Vito Volterra and Paul Lévy explored the connection between the ML function and fractional calculus (FC) (Silverman, 1972; Olver, 1997). The ML function gained renewed interest due to its applications in physics, engineering, and signal processing. Researchers introduced various generalizations, including the two-parameter ML function, three-parameter ML function and so on.

Here, the symbols N, C, R are used for set of natural numbers, set of complex numbers and set of real numbers, respectively.

The factorial function (FF) $m!$ is defined, for $m \in N \cup \{0\}$, as

$$m! = \begin{cases} \prod_{j=1}^{m} j, & m \in N, \\ 1, & m = 0. \end{cases}$$

One of its extension is called Pochhammer symbol (PS) and is defined, for $\alpha \in C$ and $n \in N \cup \{0\}$, as

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[1] UAF, Department of Math & Stat, 38900, Faisalabad, Pakistan. E-mail: ghulamfarid@uaf.edu.pk
[2] UAF, Department of Math & Stat, 38900, Faisalabad, Pakistan. Email: faizarehman176@gmail.com
[3] UAF, Department of Math & Stat, 38900, Faisalabad, Pakistan. Email: iaqsa327@gmail.com
[4] UAF, Department of Math & Stat, 38900, Faisalabad, Pakistan. Email: laraibnaeem28@gmail.com
[5] UAF, Department of Math & Stat, 38900, Faisalabad, Pakistan. Email: sitaraajmal@gmail.com
[6] UAF, Department of Math & Stat, 38900, Faisalabad, Pakistan. Email: sddddff84@gmail.com

$$(\alpha)_n = \begin{cases} 1, & n=0, \alpha \in C-\{0\}, \\ \prod_{j=1}^{n}(\alpha+j-1), & n\in N,\ \alpha\in C. \end{cases}$$

And is related with the classical FF by $(1)_n = n!$. Another generalization of the classical FF is the Gamma Function (GF) denoted as $\Gamma(\omega)$, defined for $\omega \in C, R(\omega) > 0$, as

$$\Gamma(\omega) = \int_0^\infty e^{-z} z^{\omega-1} dz,$$

which corresponds to the PS according to the relation $(\omega)_n = \dfrac{\Gamma(\omega+n)}{\Gamma(\omega)}$.

The beta function (BF) $B(\omega_1,\omega_2)$ is defined for $\omega_1,\omega_2 \in C, R(\omega_1) > 0, R(\omega_2) > 0$, as

$$B(\omega_1,\omega_2) = \int_0^1 z^{\omega_1-1}(1-z)^{\omega_2-1} dz,$$

it can be seen that BF is associated with the GF by the relation $B(\omega_1,\omega_2) = \dfrac{\Gamma(\omega_1)\Gamma(\omega_2)}{\Gamma(\omega_1+\omega_2)}$.

Diaz & Pariguan (2004) introduced $(\alpha;k)_n$, $\Gamma(\omega;k)$ and $B(\omega_1,\omega_2;k)$ be the $k-$ extensions of the classical PS, GF and BF, respectively, for $k > 0,$ as

$$(\alpha;k)_n = \begin{cases} 1, & n=0, \alpha \in C-\{0\}, \\ \prod_{j=1}^{n}(\alpha+k(j-1)), & n\in N,\ \alpha\in C, \end{cases}$$

$$\Gamma(\omega;k) = \int_0^\infty e^{-\frac{z^k}{k}} z^{\omega-1} dz$$

and

$$B(\omega_1,\omega_2;k) = \frac{1}{k}\int_0^1 z^{\frac{\omega_1}{k}-1}(1-z)^{\frac{\omega_2}{k}-1} dz;\ \omega_1,\omega_2 \in C,\ R(\omega_1) > 0, R(\omega_2) > 0.$$

One may observe that $(\alpha;1)_n = (\alpha)_n$, $\Gamma(\omega;1) = \Gamma(\omega)$ and $B(\omega_1,\omega_2;1) = B(\omega_1,\omega_2)$; moreover, it is easy to see that

$$(\alpha;k)_n = \frac{\Gamma(\alpha+nk;k)}{\Gamma(\alpha;k)},$$

$$B(\alpha,\beta;k) = \frac{\Gamma(\alpha;k)\Gamma(\beta;k)}{\Gamma(\alpha+\beta;k)} \tag{1}$$

and

$$B(\omega_1,\omega_2) = kB(\omega_1 k,\omega_2 k;k). \tag{2}$$

The ML function $M_p(z)$ in variable z with parameter p was introduced in the late 19th century as

$$M_p(z) = \sum_{m=0}^\infty \frac{1}{\Gamma(pm+1)} z^m \tag{3}$$

and has served as a cornerstone of mathematical analysis, playing a vital role in differential equations, integral transforms, and special functions. Its distinctive characteristics and adaptability have rendered it an essential instrument for addressing a broad spectrum of challenges in physics, engineering, and applied mathematics. Nevertheless, as mathematical research advances, the necessity for more comprehensive and versatile frameworks has become increasingly apparent. One may define the ML-type function $\mathfrak{M}_p(z)$ in variable z with parameter p as

$$\mathfrak{M}_p(z)=\sum_{m=0}^{\infty}\frac{1}{\Gamma(pm+1)}\frac{z^m}{m!}.$$

In response to its widespread applications, various extensions and generalizations of the classical ML function and the ML-type function have been proposed and documented in the literature. For example;

$$\mathfrak{M}_{p;r}(z)=\sum_{m=0}^{\infty}\frac{1}{\Gamma(pm+1)}\frac{z^m}{(r)_m};\ z,p\in C,R(p)>0,$$

with its $k-$ extension, for $k>0,\ p,r,z\in C;\ R(p)>0,R(r)>0$, given as

$$\mathfrak{M}_{p;r}(z;k)=\sum_{m=0}^{\infty}\frac{1}{\Gamma(pm+1;k)}\frac{z^m}{(r)_m}.$$

The two-parameter ML-type function provides a flexible tool for modeling complex systems and phenomena defined by Wiman, as

$$\mathfrak{M}_{p,q}(z)=\sum_{m=0}^{\infty}\frac{1}{\Gamma(pm+q)}\frac{z^m}{m!},$$

with its $k-$ extension, given as

$$\mathfrak{M}_{p,q}(z;k)=\sum_{m=0}^{\infty}\frac{1}{\Gamma(pm+q;k)}\frac{z^m}{m!}$$

and

$$\mathfrak{M}_{p,q;r}(z)=\sum_{m=0}^{\infty}\frac{1}{\Gamma(pm+q)}\frac{z^m}{(r)_m}$$

with its $k-$ extension, given as

$$\mathfrak{M}_{p,q;r}(z;k)=\sum_{m=0}^{\infty}\frac{1}{\Gamma(pm+q;k)}\frac{z^m}{(r)_m};\ z,p,q,r\in C;k>0,R(p)>0,R(q)>0,R(r)>0.$$

The three-parameter ML function is defined as

$$\mathfrak{M}_{d;p,q}(z)=\sum_{m=0}^{\infty}\frac{(d)_m}{\Gamma(pm+q)}\frac{z^m}{m!},$$

(Prabhakar, 1971). Some other extensions are

$$\mathfrak{M}_{d;p,q}(z;k)=\sum_{m=0}^{\infty}\frac{(d)_m}{\Gamma(pm+q;k)}\frac{z^m}{m!},$$

$$\mathfrak{M}_{d;p,q;r}(z)=\sum_{m=0}^{\infty}\frac{(d)_m}{\Gamma(pm+q)}\frac{z^m}{(r)_m},$$

$$\mathfrak{M}_{d;p,q;r}(z;k)=\sum_{m=0}^{\infty}\frac{(d)_m}{\Gamma(pm+q;k)}\frac{z^m}{(r)_m},$$

$$\mathfrak{M}_{d,i;p,q;r,j}(z)=\sum_{m=0}^{\infty}\frac{(d)_{im}}{\Gamma(pm+q)}\frac{z^m}{(r)_{jm}} \tag{4}$$

and

$$_{\rho}\mathfrak{M}_{d,i;p,q;r,j}(z;\sigma)=\sum_{m=0}^{\infty}\frac{B_{\sigma}(\rho+mi,d-\rho)}{B(\rho,d-\rho)}\frac{(d)_{mi}}{\Gamma(pm+q)}\frac{z^m}{(r)_{mj}} \tag{5}$$

with $d, p, q, r, \rho, z \in C; \min\{R(d)>0, R(p)>0, R(q)>0, R(r)>0\}; \quad \sigma, i, j>0$ and $i \leq R(p)+j$ (Salim & Faraj, 2012; Andrić et al., 2018).

The FI, a mathematical operation that generalizes the traditional integer-order integral, has a rich history and significant importance in various fields. The concept of FC, including FIs, was first explored by Gottfried Wilhelm Leibniz and Guillaume François Antoine, Marquis de L'Hôpital in 1695. FIs help to describe complex systems with memory, non-locality, and power-law behavior, which are common in physics, engineering, biology, and finance (Baleanu, 2012; Hilfer, 2000; Sabatier et al., 2007). The RL fractional integral is a type of FI that generalizes the traditional Riemann integral to fractional orders. It's named after Bernhard Riemann and Joseph Liouville, who contributed to its development. The left and right RL fractional integral of order $\alpha>0$ of a function $\zeta(t)$ are defined as

$$ {}_{a+}^{RL}\Upsilon_t^{\alpha}\{\zeta(t)\} = \frac{1}{\Gamma(\alpha)}\int_a^t (t-z)^{\alpha-1}\zeta(z)dz $$

and

$$ {}_{b-}^{RL}\Upsilon_t^{\alpha}\{\zeta(t)\} = \frac{1}{\Gamma(\alpha)}\int_t^b (z-t)^{\alpha-1}\zeta(z)dz, $$

respectively (Hilfer, 2000).

The left and the right RL $k-$FIs of $\zeta(t)$ are defined as

$$ {}_{a+}^{RL}\Upsilon_t^{\alpha}\{\zeta(t);k\} = \frac{1}{k\Gamma(\alpha;k)}\int_a^t (t-z)^{\frac{\alpha}{k}-1}\zeta(z)dz \tag{6} $$

and

$$ {}_{b-}^{RL}\Upsilon_t^{\alpha}\{\zeta(t);k\} = \frac{1}{k\Gamma(\alpha;k)}\int_t^b (z-t)^{\frac{\alpha}{k}-1}\zeta(z)dz \tag{7} $$

(Mobeen & Habibullah, 2012).

Remarks: One may observe that

- $\mathfrak{M}_p(z;1) = \mathfrak{M}_p(z)$
- $\mathfrak{M}_{p;1}(z) = \mathfrak{M}_p(z)$
- $\mathfrak{M}_{d;p,q;1}(z) = \mathfrak{M}_{d;p,q}(z)$
- $\mathfrak{M}_{d;p,q}(z;1) = \mathfrak{M}_{d;p,q}(z)$
- $\mathfrak{M}_{d;p,q;1}(z;1) = \mathfrak{M}_{d;p,q}(z)$
- ${}_{\rho}\mathfrak{M}_{d,i;p,q;r,j}(z;0) = {}_{\rho}\mathfrak{M}_{d,i;p,q;r,j}(z)$

2. Literature Review

The ML function is a special function that plays a crucial role in various areas of mathematics, physics, and engineering due to its unique properties and wide range of applications like as, MLF is a fundamental solution to fractional differential equations, which are used to model complex systems with memory and non-locality, MLF describes anomalous diffusion processes, which are common in complex systems like porous media, biological tissues, and disordered materials, MLF is used to model viscoelastic behavior in materials, which is essential for understanding stress-strain relationships in complex materials, it is applied in electrical circuit analysis, signal processing, and control theory any many other interdisciplinary areas (Silverman, 1972; Kanemitsu & Tsukada, 2007).

Generalized ML function were introduced (Wiman, 1905). A singular integral equation with the generalized ML function in the kernel was presented (Prabhakar, 1971). The solutions to fractional-order equations were found associated with ML-type functions including the second-kind Abel integral equations, significant in typical inverse problems, as well as fractional differential equations (Mainardi & Gorenflo, 2000). Some FC operators were applied on several generalized forms of the ML functions (Kilbas et al., 2004).

The $k-$generalized GF, BF and the pochhammer $k-$symbol were introduced. Various identities that extend those associated with the classical GF, BF, and PS were established and integral representations for these representations were presented (Diaz & Pariguan, 2004). A new type of FI, referred to as the $k-$FI, which is based on the $k-$GF and its applications were presented (Mubeen & Habibullah, 2012).

A newly generalized ML-type function was introduced. Various properties of this function, including differentiation, the Laplace transform, the BF, the Mellin transform, the Whittaker transform, its generalized hypergeometric series form, the Mellin-Barnes integral representation, and its relationship with Fox's H-function and Wright's hypergeometric function were presented (Salim & Faraj, 2012).

The Montgomery identities were generalized to apply to the RL FIs. Utilizing these extended Montgomery identities several new integral inequalities were derived (Sarikaya & Ogunmez, 2012). Some new FI inequalities were established (Tariboon et al., 2014). The extended ML functions by utilizing the extended BF was introduced and several integral representations were derived. The Mellin transform of these functions was expressed in terms of generalized Wright hypergeometric functions. Lastly, some connections between these functions and the Laguerre polynomials and Whittaker functions were provided (Özarslan & Yılmaz, 2014).

An extended generalized ML function and the associated FI operator were utilized to derive generalizations of Opial-type inequalities originally established by mitrinović and pečarić. Additionally, various interesting properties of this function and its integral operator were explored. Several known results are also derived as special cases (Andrić et al., 2018). The limits of RL FIs using $(h-m)-$convex functions were investigated and upper bounds for the sum of left and right fractional integral for $(h-m)-$convex functions were determined. Additionally, a hadamard type inequality was derived under an extra condition (Farid, 2021).

3. Materials and Methods

The generalized form of the classical BF $B_\sigma(u,v)$ is defined as

$$B_\sigma(u,v)=\int_0^1 z^{u-1}(1-z)^{v-1}e^{-\frac{\sigma}{z(1-z)}}dz \tag{8}$$

(Ozarslan & Yilmaz, 2014).

We introduce $k-$extension of the generalized classical beta function, defined above, as

$$B_\sigma(u,v;k)=\frac{1}{k}\int_0^1 z^{\frac{u}{k}-1}(1-z)^{\frac{v}{k}-1}e^{-\frac{\sigma}{z(1-z)}}dz. \tag{9}$$

Moreover, we define triple$-k$ extension of (4) and penta$-k$ extension of (5), respectively, as

$$\mathfrak{M}_{d,i;p,q;r,j}(z;k_1;k,k_2)=\sum_{m=0}^{\infty}\frac{(d;k_1)_{mi}}{\Gamma(pm+q;k)}\frac{z^m}{(r;k_2)_{mj}} \tag{10}$$

and

$$\begin{aligned}&{}_{\rho}\mathfrak{M}_{d,i;p,q;r,j}(z;k_1,k_3;k_2,k,k_4;\sigma)\\&=\sum_{m=0}^{\infty}\frac{B_{\sigma}(\rho+mi,d-\rho;k_1)}{B(\rho,d-\rho;k_2)}\frac{(d;k_3)_{mi}}{\Gamma(pm+q;k)}\frac{z^m}{(r;k_4)_{mj}},\end{aligned} \tag{11}$$

with positive k and k_i for each $i\in\{1,2,3,4\}$.

Proposition:
Suppose there exists a function f for which the following integral is defined, and let $a\in R$. Then, for each x in (a,t), if

$$\Upsilon_{a;p,q}\{f(t)\}=\int_a^t (f(t)-f(x))^p(f(x))^q d(f(x)),$$

then we get

$$\Upsilon_{a;p,q}\{f(t)\}=(f(t))^{p+q+1}B(p+1,q+1). \tag{12}$$

Proof: The result is a consequence of the substitution $y=\dfrac{f(x)}{f(t)}$.

Corollary 1: $\Upsilon_{a;p,q}(t)=(t)^{p+q+1}B(p+1,q+1).$

Corollary 2: By the use of (2), corollary 1 leads to $\Upsilon_{a;p,q}(t)=(t)^{p+q+1}kB((p+1)k,(q+1)k;k).$

For $p=\dfrac{\alpha}{k}-1,\ q=\dfrac{pm+q}{k}-1$, (12) becomes

$$\Upsilon_{a;\frac{\alpha}{k}-1,\frac{pm+q}{k}-1}(t)=k(t)^{\frac{pm+\alpha+q}{k}+1}B(\alpha,pm+q;k). \tag{13}$$

4. Results and Discussions

The ML function is crucial in the RL FI because by engaging the ML function with the RL $k-$fractional integral, researchers can tackle complex problems, gain deeper insights and develop innovative solutions. Here, we use the values $\zeta_0(t)$ and $\zeta_1(t)$ of $\zeta(t)$, given by $\zeta_0(t)=t^{\frac{q}{k}-1}\mathfrak{M}_{d,i;p,q;r,j}((ct)^{\frac{p}{k}};k_1;k,k_2)$ and $\zeta_1(t)=t^{\frac{q}{k}-1}{}_{\rho}\mathfrak{M}_{d,i;p,q;r,j}((ct)^{\frac{p}{k}};k_1,k_3;k_2,k,k_4;\sigma)$; we want to find ${}^{RL}_{a+}\Upsilon^{\alpha}_{t}\{\zeta(t);k\}$ and ${}^{RL}_{b-}\Upsilon^{\alpha}_{t}\{\zeta(t);k\}$ for each of $\zeta(t)=\zeta_0(t)$ and for $\zeta(t)=\zeta_1(t)$.

Theorem 1

$${}^{RL}_{0+}\Upsilon^{\alpha}_{t}\{t^{\frac{q}{k}-1}\mathfrak{M}_{d,i;p,q;r,j}((ct)^{\frac{p}{k}};k_1;k,k_2);k\}=t^{\frac{\alpha+q}{k}-1}\mathfrak{M}_{d,i;p,q+\alpha;r,j}((ct)^{\frac{p}{k}};k_1;k,k_2).$$

Proof:
Using Eq. (6), we have

$$ {}^{RL}_{a+}\Upsilon_t^{\alpha}\{\zeta_0(t);k\} = \frac{1}{k\Gamma(\alpha;k)}\int_a^t (t-z)^{\frac{\alpha}{k}-1}\zeta_0(z)dz $$
$$ = \frac{1}{k\Gamma(\alpha;k)}\int_a^t (t-z)^{\frac{\alpha}{k}-1} z^{\frac{q}{k}-1}\mathfrak{M}_{d,i;p,q;r,j}((cz)^{\frac{p}{k}};k_1;k,k_2)dz. $$

By using (10), it follows that

$$ {}^{RL}_{a+}\Upsilon_t^{\alpha}\{\zeta_0(t);k\} = \frac{1}{k\Gamma(\alpha;k)}\int_a^t (t-z)^{\frac{\alpha}{k}-1} z^{\frac{q}{k}-1}\sum_{m=0}^{\infty}\frac{(d;k_1)_{im}}{\Gamma(pm+q;k)}\frac{((cz)^{\frac{p}{k}})^m}{(r;k_2)_{jm}}dz $$
$$ = \frac{1}{k\Gamma(\alpha;k)}\sum_{m=0}^{\infty}\frac{(d;k_1)_{im}(c)^{\frac{pm}{k}}}{\Gamma(pm+q;k)}\frac{1}{(r;k_2)_{jm}}\int_a^t (t-z)^{\frac{\alpha}{k}-1} z^{\frac{pm}{k}+\frac{q}{k}-1}dz $$
$$ = \frac{1}{k\Gamma(\alpha;k)}\sum_{m=0}^{\infty}\frac{(d;k_1)_{im}(c)^{\frac{pm}{k}}}{\Gamma(pm+q;k)}\frac{1}{(r;k_2)_{jm}}\Upsilon_{a,\frac{\alpha}{k}-1,\frac{pm+q}{k}-1}(t). $$

With use of (13) for $a=0$, the above equation leads to

$$ {}^{RL}_{0+}\Upsilon_t^{\alpha}\{\zeta_0(t);k\} = \frac{1}{k\Gamma(\alpha;k)}\sum_{m=0}^{\infty}\frac{(d;k_1)_{im}(c)^{\frac{pm}{k}}}{\Gamma(pm+q;k)}\frac{1}{(r;k_2)_{jm}}k(t)^{\frac{pm+\alpha+q}{k}+1}B(\alpha,pm+q;k) $$
$$ = \frac{1}{\Gamma(\alpha;k)}\sum_{m=0}^{\infty}\frac{(d;k_1)_{im}(c)^{\frac{pm}{k}}}{\Gamma(pm+q;k)}\frac{1}{(r;k_2)_{jm}}t^{\frac{\alpha+pm+q}{k}-1}B(pm+q,\alpha;k). $$

It, by use of (1), takes the form

$$ {}^{RL}_{0+}\Upsilon_t^{\alpha}\{\zeta_0(t);k\} = \frac{1}{\Gamma(\alpha;k)}\sum_{m=0}^{\infty}\frac{(d;k_1)_{im}(c)^{\frac{pm}{k}}}{\Gamma(pm+q;k)}\frac{1}{(r;k_2)_{jm}}t^{\frac{\alpha+pm+q}{k}-1}\frac{\Gamma(pm+q;k)\Gamma(\alpha;k)}{\Gamma(pm+q+\alpha;k)} $$
$$ = t^{\frac{\alpha+q}{k}-1}\sum_{m=0}^{\infty}\frac{(d;k_1)_{im}(c)^{\frac{pm}{k}}}{\Gamma(pm+q+\alpha;k)}\frac{(t^{\frac{p}{k}})^m}{(r;k_2)_{jm}} $$
$$ = t^{\frac{\alpha+q}{k}-1}\sum_{m=0}^{\infty}\frac{(d;k_1)_{im}}{\Gamma(pm+q+\alpha;k)}\frac{(ct^{\frac{p}{k}})^m}{(r;k_2)_{jm}}. $$

With use of Eq. (10), it becomes

$$ {}^{RL}_{0+}\Upsilon_t^{\alpha}\{\zeta_0(t);k\} = t^{\frac{\alpha+q}{k}-1}\mathfrak{M}_{d,i;p,q+\alpha;r,j}((ct)^{\frac{p}{k}};k_1;k,k_2). $$

Theorem 2

$$ {}^{RL}_{0+}\Upsilon_t^{\alpha}\{t^{\frac{q}{k}-1}{}_{\lambda}\mathfrak{M}_{d,i;p,q;r,j}((ct)^{\frac{p}{k}};k_1,k_3;k_2,k,k_4;\sigma);k\} $$
$$ = t^{\frac{\alpha+q}{k}-1}{}_{\rho}\mathfrak{M}_{d,i;p,q+\alpha;r,j}((ct)^{\frac{p}{k}};k_1,k_3;k_2,k,k_4;\sigma). $$

Proof:

Using (6), (11), (13) and (1); the result follows as

$$ {}^{RL}_{a+}\Upsilon_t^{\alpha}\{\zeta_1(t);k\} = \frac{1}{k\Gamma(\alpha;k)}\int_a^t (t-z)^{\frac{\alpha}{k}-1}\zeta_1(z)dz $$

$$=\frac{1}{k\Gamma(\alpha;k)}\int_a^t (t-z)^{\frac{\alpha}{k}-1} z^{\frac{q}{k}-1}\,{}_{\rho}\mathfrak{M}_{d,i;p,q;r,j}((cz)^{\frac{p}{k}};k_1,k_3;k_2,k,k_4;\sigma)dz$$

$$=\frac{1}{k\Gamma(\alpha;k)}\int_a^t (t-z)^{\frac{\alpha}{k}-1} z^{\frac{q}{k}-1}\sum_{m=0}^{\infty}\frac{B_{\sigma}(\rho+mi,d-\rho;k_1)}{B\ (\rho,d-\rho;k_2)}\frac{(d;k_3)_{mi}}{\Gamma(pm+q;k)}\frac{((cz)^{\frac{p}{k}})^m}{(r;k_4)_{mj}}dz$$

$$=\frac{1}{k\Gamma(\alpha;k)}\sum_{m=0}^{\infty}\frac{B_{\sigma}(\rho+mi,d-\rho;k_1)}{B(\rho,d-\rho;k_2)}\frac{(d;k_3)_{mi}(c)^{\frac{pm}{k}}}{\Gamma(pm+q;k)}\frac{1}{(r;k_4)_{mj}}\int_a^t (t-z)^{\frac{\alpha}{k}-1} z^{\frac{pm}{k}+\frac{q}{k}-1}dz$$

$$=\frac{1}{k\Gamma(\alpha;k)}\sum_{m=0}^{\infty}\frac{B_{\sigma}(\rho+mi,d-\rho;k_1)}{B\ (\rho,d-\rho;k_2)}\frac{(d;k_3)_{mi}(c)^{\frac{pm}{k}}}{\Gamma(pm+q;k)}\frac{1}{(r;k_4)_{mj}}\Upsilon_{a,\frac{\alpha}{k}-1,\frac{pm+q}{k}-1}(t).$$

For $a=0$; we have

$${}^{RL}_{0+}\Upsilon^{\alpha}_{t}\{\zeta_1(t);k\}$$

$$=\frac{1}{k\Gamma(\alpha;k)}\sum_{m=0}^{\infty}\frac{B_{\sigma}(\rho+mi,d-\rho;k_1)}{B(\rho,d-\rho;k_2)}\frac{(d;k_3)_{mi}(c)^{\frac{pm}{k}}}{\Gamma(pm+q;k)}\frac{1}{(r;k_4)_{mj}}k(t)^{\frac{pm+\alpha+q}{k}+1}B(\alpha,pm+q;k)$$

$$=\frac{1}{\Gamma(\alpha;k)}\sum_{m=0}^{\infty}\frac{B_{\sigma}(\rho+mi,d-\rho;k_1)}{B(\rho,d-\rho;k_2)}\frac{(d;k_3)_{mi}(c)^{\frac{pm}{k}}}{\Gamma(pm+q;k)}\frac{1}{(r;k_4)_{mj}}t^{\frac{\alpha+pm+q}{k}-1}\frac{\Gamma(pm+q;k)\Gamma(\alpha;k)}{\Gamma(pm+q+\alpha;k)}$$

$$=t^{\frac{\alpha+q}{k}-1}\sum_{m=0}^{\infty}\frac{B_{\sigma}(\rho+mi,d-\rho;k_1)}{B(\rho,d-\rho;k_2)}\frac{(d;k_3)_{mi}}{\Gamma(pm+q+\alpha;k)}\frac{((ct)^{\frac{p}{k}})^m}{(r;k_4)_{mj}}$$

$$=t^{\frac{\alpha+q}{k}-1}\,{}_{\rho}\mathfrak{M}_{d,i;p,q+\alpha;r,j}((ct)^{\frac{p}{k}};k_1,k_3;k_2,k,k_4;\sigma).$$

By extending the same line of reasoning and derivation, we arrive at the following results.

Theorem 3

$${}^{RL}_{0-}\Upsilon^{\alpha}_{t}\{t^{\frac{q}{k}-1}\mathfrak{M}_{d,i;p,q;r,j}((ct)^{\frac{p}{k}};k_1;k,k_2);k\}=(-1)^{\frac{\alpha}{k}}t^{\frac{\alpha+q}{k}-1}\mathfrak{M}_{d,i;p,q+\alpha;r,j}((ct)^{\frac{p}{k}};k_1;k,k_2).$$

Theorem 4

$${}^{RL}_{0-}\Upsilon^{\alpha}_{t}\{t^{\frac{q}{k}-1}\,{}_{\rho}\mathfrak{M}_{d,i;p,q;r,j}((ct)^{\frac{p}{k}};k_1,k_3;k_2,k,k_4;\sigma);k\}$$
$$=(-1)^{\frac{\alpha}{k}}t^{\frac{\alpha+q}{k}-1}\,{}_{\rho}\mathfrak{M}_{d,i;p,q+\alpha;r,j}((ct)^{\frac{p}{k}};k_1,k_3;k_2,k,k_4;\sigma).$$

In current study, $k-$extension of generalized BF, triple$-k$ extension and penta$-k$ extension of generalized ML are introduced. While using these $k-$extensions RL $k-$FIs are evaluated.

Since the MLFs defined from (3) to (5) are all specific cases of the extended forms presented in (10) and (11), the results established in Theorem 1 to Theorem 4 obviously follow particularly for all the specific cases as well.

5. Conclusion

In this comprehensive investigation, we introduce and explain the $k-$extension of various generalized ML functions, thereby broadening the scope of these fundamental functions. Subsequently, we connect these $k-$extension in conjunction with the RL $k-$fractional integral, a powerful tool in FC. By integrating these $k-$extension into the RL fractional integrals, one can create a novel framework for analyzing and solving complex problems in diverse fields such as physics, engineering, and applied mathematics. This study contributes to the advancement of mathematical analysis and its applications may be used in modeling real-world phenomena.

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