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# Triple-*k* And Penta- *k* Extended Forms Of Some Generalized Forms Of Mittag-Leffler Functions And Their Hadamard *k* -Fractional Integrals

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**Abstract.**
*In this work, a $k-$extended form of a generalized form of the classical beta function is introduced. Moreover; a triple $-k$ extended form and a penta $-k$ extended form, based on the extended beta function, of some generalized forms of the classical Mittag-Leffler function, are introduced. Furthermore, the Hadamard $k-$fractional integrals of some functions involving these extended Mittag-Lefflers functions are found.*

***Keywords:*** *Mittag-Leffler function, Beta function, $k-$Hadamard fractional intergral.*

## 1. Introduction

In this article, the notation $a^{(k)}$ stands for $a$ with parameter $k>0$, however if particularly $k \in Z^+$, then $a^{(k)}$ will represent $a_k$ which is in fact $k^{th}$ term of the sequence $(a_n)_{n=1}^{\infty}$. The notation $\mathrm{Re}(z)$ is used for real part of the complex number $z$. Moreover, the notations $Z^+ = N$, $Z^-$ and $C$ are used for the set of positive integers (the set of natural numbers), the set of negative integers and the set of complex numbers, respectively.

Furthermore, we use the abbreviations HF, DE, IE, IDE, PS, FF, GF, BF, MLF, FC, FD, FI, RL, GL, EK and LC for Hypergeometric Function, Differential Equation, Integral Equation, Integro-differential Equation, Pochhammer symbol, factorial function, gamma function, beta function, Mittag-Leffler function, fractional calculus, fractional derivative, fractional integral, Riemann-Liouville, Grünwald-Letnikov, Erdélyi-Kober and Liouville-Caputo, respectively.

The classical FF $n!$ is defined, for $n \in Z^+ \cup \{0\}$, as

$$n! = \begin{cases} \prod_{k=1}^{n} k, & n \in Z^+, \\ 1, & n = 0. \end{cases}$$

One of its generalized form is the PS and is defined, for $\alpha \in C$ and $n \in Z^+ \cup \{0\}$, as

$$(\alpha)_n = \begin{cases} 1, & n = 0, \alpha \in C - \{0\}, \\ \prod_{k=1}^{n} (\alpha + k - 1), & n \in N,\ \alpha \in C \end{cases}$$

and is related with the classical FF by $(1)_n = n!$.

Another extension of the classical FF is the GF $\Gamma(\alpha)$, defined for $\alpha \in C - (Z^- \cup \{0\})$, as

$$\Gamma(\alpha) = \int_0^{\infty} e^{-w} w^{\alpha-1} dw,$$

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which is related with the PS by the relation $(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$. The beta function $B(\alpha,\beta)$ is defined, for $\alpha,\beta \in C-(Z^- \cup \{0\})$, as

$$B(\alpha,\beta) = \int_0^1 w^{\alpha-1}(1-w)^{\beta-1}dw,$$

which is related with the GF by the relation $B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.

Some generalized forms called Pochhammer $k-$symbol $(\alpha)_n^{(k)}$, $k-$GF $\Gamma^{(k)}(\alpha)$ and $k-$BF $B^{(k)}(\alpha,\beta)$, of the related classical functions are defined as

$$(\alpha)_n^{(k)} = \begin{cases} 1, & n=0, \alpha \in C-\{0\}, \\ \prod_{j=1}^{n}(\alpha+k(j-1)), & n \in N,\ \alpha \in C, \end{cases}$$

$$\Gamma^{(k)}(\alpha) = \int_0^\infty e^{-\frac{w^k}{k}} w^{\alpha-1}dw$$

and

$$B^{(k)}(\alpha,\beta) = \frac{1}{k}\int_0^1 w^{\frac{\alpha}{k}-1}(1-w)^{\frac{\beta}{k}-1}dw.$$

The generalized functions are related with each other, in an analogous way to that of the classical functions, by

$$(\alpha)_n^{(k)} = \frac{\Gamma^{(k)}(\alpha+nk)}{\Gamma^{(k)}(\alpha)}$$

and

$$B^{(k)}(\alpha,\beta) = \frac{\Gamma^{(k)}(\alpha)\Gamma^{(k)}(\beta)}{\Gamma^{(k)}(\alpha+\beta)}; \tag{1}$$

moreover, it is easy to see that

$$B(\alpha,\beta) = kB^{(k)}(\alpha k,\beta k) \tag{2}$$

(Diaz & Pariguan, 2004). Another generalized form of the classical BF is $B_\sigma$ and is defined as

$$B_\sigma(p,q) = \int_0^1 w^{p-1}(1-w)^{q-1}e^{-\frac{\sigma}{w(1-w)}}dw \tag{3}$$

(Özarslan & Yılmaz, 2014).

The ML function $L_p(z)$ in variable $z$ with parameter $p$ was introduced in the late 19th century as

$$L_p(z) = \sum_{m=0}^{\infty}\frac{1}{\Gamma(pm+1)}z^m$$

and has been a foundation of mathematical analysis, particularly in the realms of differential equations, integral transforms, and special functions. Its unique properties and versatility have made it an indispensable tool for solving a wide range of problems in physics, engineering, and applied mathematics. However, as research continues to push the boundaries of mathematical knowledge, the need for more general and flexible frameworks has become increasingly evident.

One may also define the ML-type function $M_p(z)$ in variable $z$ with parameter $p$ as

$$M_p(z) = \sum_{m=0}^{\infty}\frac{1}{\Gamma(pm+1)}\frac{z^m}{m!}.$$

A number of extension/generalizations of the classical ML and the ML-type functions have been introduced and are available in the literature. One may list some of these as

$$M_{p,q}(z)=\sum_{m=0}^{\infty}\frac{1}{\Gamma(pm+q)}\frac{z^m}{m!},$$
$$M_{d;p,q}(z)=\sum_{m=0}^{\infty}\frac{(d)_m}{\Gamma(pm+q)}\frac{z^m}{m!},$$
$$M_{p;r}(z)=\sum_{m=0}^{\infty}\frac{1}{\Gamma(pm+1)}\frac{z^m}{(r)_m},$$
$$M_{p,q;r}(z)=\sum_{m=0}^{\infty}\frac{1}{\Gamma(pm+q)}\frac{z^m}{(r)_m},$$
$$M_{d;p,q;r}(z)=\sum_{m=0}^{\infty}\frac{(d)_m}{\Gamma(pm+q)}\frac{z^m}{(r)_m},$$
$$M_{p}^{(k)}(z)=\sum_{m=0}^{\infty}\frac{1}{\Gamma^{(k)}(pm+1)}z^m,$$
$$M_{p,q}^{(k)}(z)=\sum_{m=0}^{\infty}\frac{1}{\Gamma^{(k)}(pm+q)}z^m,$$
$$M_{d;p,q}^{(k)}(z)=\sum_{m=0}^{\infty}\frac{(d)_m}{\Gamma^{(k)}(pm+q)}\frac{z^m}{m!},$$
$$M_{p;r}^{(k)}(z)=\sum_{m=0}^{\infty}\frac{1}{\Gamma^{(k)}(pm+1)}\frac{z^m}{(r)_m},$$
$$M_{p,q;r}^{(k)}(z)=\sum_{m=0}^{\infty}\frac{1}{\Gamma^{(k)}(pm+q)}\frac{z^m}{(r)_m},$$
$$M_{d;p,q;r}^{(k)}(z)=\sum_{m=0}^{\infty}\frac{(d)_m}{\Gamma^{(k)}(pm+q)}\frac{z^m}{(r)_m},$$
$$M_{d,i;p,q;r,j}(z)=\sum_{m=0}^{\infty}\frac{(d)_{im}}{\Gamma(pm+q)}\frac{z^m}{(r)_{jm}} \tag{4}$$

and

$$_{\lambda}M_{d,i;p,q;r,j}(z;\sigma)=\sum_{m=0}^{\infty}\frac{B_{\sigma}(\lambda+mi,d-\lambda)}{B(\lambda,d-\lambda)}\frac{(d)_{mi}}{\Gamma(pm+q)}\frac{z^m}{(r)_{mj}}. \tag{5}$$

In all of these definitions: $p,q,d,r,z\in C$, $\min\{\mathrm{Re}(p),\ \mathrm{Re}(q),\ \mathrm{Re}(d),\ \mathrm{Re}(r)\}>0$ and $i,j,\sigma>0$ with $i\leq \mathrm{Re}(p)+j$ (Özarslan & Yılmaz, 2014). To see the details of the conditions on the parameters used, one may see Andric et al. (2018). For the study of special functions, we refer (Silverman, 1972; Andrews et al., 1999).

FC is a branch of mathematical analysis that extends the concepts of classical calculus to fractional-order derivatives and integrals. It provides a framework for modeling complex phenomena in various fields, including physics, engineering, and finance, where traditional integer-order calculus is insufficient. This enables the description of non-local, non-integer, and scale-invariant properties of systems, making it a powerful tool for modeling like Memory-dependent processes, Non-Markovian systems, Fractal and multifractal behaviors and Anomalous diffusion and transport.

The mathematical foundations of fractional calculus date back to the 17th century, but its applications have grown significantly in recent decades. Today, fractional calculus is used to model and analyze complex systems in various disciplines, including: Signal processing, Image analysis, Control theory, Biomedical engineering and Econophysics.

Some of FDs available in the literature include: RL derivative, Caputo derivative, GL derivative, Weyl derivative, Hadamard derivative, EK derivative, LC derivative, Marchaud derivative, Riesz derivative and Hilfer derivative. Likewise, some of the FIs available in the literature involve RL Integral, Caputo integral, Hadamard Integral, EK Integral, Weyl Integral, Riesz Integral, Feller integral, Saigo integral, Kober integral and Katugampola Integral.

For $a<t<b$; the left and the right Hadamard FIs of $\xi(t)$ are defined as

$$ {}_{a+}^{H}I_t^{\alpha}\{\xi(t)\} = \frac{1}{\Gamma(\alpha)}\int_a^t (\ln t - \ln w)^{\alpha-1}\frac{\xi(w)}{w}dw $$

and

$$ {}_{b-}^{H}I_t^{\alpha}\{\xi(t)\} = \frac{1}{\Gamma(\alpha)}\int_t^b (\ln w - \ln t)^{\alpha-1}\frac{\xi(w)}{w}dw, $$

respectively (Kilbas, 2001; Farid & Habibullah, 2015).

The left and the right Hadamard $k-$FIs of $\xi(t)$ are defined as

$$ {}_{a+}^{H}I_t^{\alpha;k}\{\xi(t)\} = \frac{1}{k\Gamma^{(k)}(\alpha)}\int_a^t (\ln t - \ln w)^{\frac{\alpha}{k}-1}\frac{\xi(w)}{w}dw \tag{6} $$

and

$$ {}_{b-}^{H}I_t^{\alpha;k}\{\xi(t)\} = \frac{1}{k\Gamma^{(k)}(\alpha)}\int_t^b (\ln w - \ln t)^{\frac{\alpha}{k}-1}\frac{\xi(w)}{w}dw, \tag{7} $$

respectively (Farid & Habibullah, 2015). For the study of topics relevant to FC; we refer (Das, 2011; Hilfer, 2000).

## 2. Literature Review

Mainardi & Gorenflo (2000) studied solutions of FD and FI equations involving Abel integral equations of the second kind, in terms of some ML type functions. Kilbas (2001) studied fractional integration and fractional differentiation on finite intervals with Hadamard settings and thus generalized the idea of differentiation and integration; moreover, the boundedness of the operators was also studied. Diaz and Pariguan (2004) introduced a number of k-extended special functions involving k-PS, k-GF and k-BF and established several results related with the k-extended special functions. Kilbas et al. (2004) applied some fractional calculus operators on several generalized forms of the Mittag-Leffler functions.

Haubold et al. (2011) studied some properties of the Mittag-Leffler function and its generalized forms; moreover a number of applications of the generalized forms of the Mittag-Leffler functions in several areas of science and engineering were also presented. Purohit et al. (2011) studied several properties of the multi-index ML functions and established certain theorems related with Saigo FI operators. Rida & Arafa (2011) developed some new applications of the ML functions to find solutions of some linear fractional differential equations; furthermore, the fractional derivatives were described in the Caputo sense and to ensure the reliability of the method, several examples were considered.

Dorrego & Cerutti (2012) found solutions of some fractional DEs and fractional IDEs by introducing some k-extended forms of the classical ML functions; moreover, some basic properties of the extended ML functions were also studied. Salim & Faraj (2012) introduced a new generalized form of ML type function and studied its properties involving: differentiation, transforms, generalized hypergeometric series form, integral representation and its relationship with some special functions. Özarslan & Yılmaz (2014) presented the extended ML functions by using the extended BFs and obtained some integral representations of them; furthermore, the Mellin transform of these functions were also studied in terms of generalized Wright HFs.

Faird & Habibullah (2015) introduced an extended form called Hadamard k-fractional integral and discussed some basic properties involving semigroup property, commutative property and boundedness of the extended integral operator. Ma & Li (2017) investigated three aspects of Hadamard FC; semigroup property, definite conditions related with Hadamard type fractional operators and Gronwall inequality with weak singularity; moreover, some examples for the illustration were also discussed.

Andric et al. (2018) defined an extended generalized ML function and the related FI; moreover, some properties of the extended function and the related fractional integral inequalities were also discussed. Srivastava et al. (2018) introduced and investigated some FI operators involving some generalized multi-index ML functions; moreover, a number of interesting results for the composition of such well-known FD and FI operators were presented. Padma et al. (2023) introduced extended generalized ML function through the extended BF and obtained certain integral and differential representation; moreover, some formulae related with RL fractional integration and differentiation operators.

## 3. Materials and Methods

We define $k-$ extended form $B_{\sigma}^{(k)}(p,q)$ of a generalized form of the classical beta function $B_{\sigma}(p,q)$ defined in (3), as

$$B_{\sigma}^{(k)}(p,q)=\frac{1}{k}\int_0^1 w^{\frac{p}{k}-1}(1-w)^{\frac{q}{k}-1}e^{-\frac{\sigma}{w(1-w)}}dw. \tag{8}$$

Moreover, we define the triple$-k$ extended form of (4) and penta$-k$ extended form of (5), as

$$M_{d,i;p,q;r,j}^{(k_1;k,k_2)}(z)=\sum_{m=0}^{\infty}\frac{(d)_{im}^{(k_1)}}{\Gamma^{(k)}(pm+q)}\frac{z^m}{(r)_{jm}^{(k_2)}} \tag{9}$$

and

$${}_{\lambda}M_{d,i;p,q;r,j}^{(k_1,k_3;k_2,k,k_4)}(z;\sigma)=\sum_{m=0}^{\infty}\frac{B_{\sigma}^{(k_1)}(\lambda+mi,d-\lambda)}{B^{(k_2)}(\lambda,d-\lambda)}\frac{(d)_{mi}^{(k_3)}}{\Gamma^{(k)}(pm+q)}\frac{z^m}{(r)_{mj}^{(k_4)}} \tag{10}$$

respectively. Here, $k,k_i>0$ for each $i\in\{1,2,3,4\}$.

**Lemma:** Let $f$ be a suitable function for which the following integral exists and $a$ be a real number for which $f(a)=0$. Then for each $x\in]a,t[$, if

$$I_{a;p,q}^{(f)}(t)=\int_a^t (f(t)-f(x))^p(f(x))^q d(f(x)),$$

then we have

$$I_{a;p,q}^{(f)}(t)=(f(t))^{p+q+1}B(p+1,q+1). \tag{11}$$

**Proof:** The results follows due to the substitution $y=\dfrac{f(x)}{f(t)}$.

**Corollary 1:** $I_{a;p,q}^{(\ln)}(t)=(\ln(t))^{p+q+1}B(p+1,q+1).$

**Corollary 2:** Using (2), it follows that $I_{a;p,q}^{(\ln)}(t)=(\ln(t))^{p+q+1}kB^{(k)}((p+1)k,(q+1)k).$

Hence, (11) leads to

$$I_{a;\frac{\alpha}{k}-1,\frac{pm+q}{k}-1}^{(\ln)}(t)=k(\ln(t))^{\frac{pm+\alpha+q}{k}+1}B^{(k)}(\alpha,pm+q). \tag{12}$$

## 4. Results and Discussion

### 4.1 Results

Now we use the values $\xi_0(t)$ and $\xi_1(t)$ of $\xi(t)$, given by

$$\xi_0(t)=(\ln t)^{\frac{q}{k}-1}M_{d,i;p,q;r,j}^{(k_1;k,k_2)}((\ln t^c)^{\frac{p}{k}}) \tag{13}$$

and

$$\xi_1(t)=(\ln t)^{\frac{q}{k}-1}{}_{\lambda}M_{d,i;p,q;r,j}^{(k_1,k_3;k_2,k,k_4)}((\ln t^c)^{\frac{p}{k}};\sigma); \tag{14}$$

we want to find ${}_{a+}^{\;H}I_t^{\alpha;k}\{\xi(t)\}$ and ${}_{b-}^{\;H}I_t^{\alpha;k}\{\xi(t)\}$ for each of $\xi(t)=\xi_0(t)$ and for $\xi(t)=\xi_1(t)$.

**Theorem 1:**

$${}_{1+}^{\;H}I_t^{\alpha;k}\{(\ln t)^{\frac{q}{k}-1}M_{d,i;p,q;r,j}^{(k_1;k,k_2)}((\ln t^c)^{\frac{p}{k}})\}=(\ln t)^{\frac{\alpha+q}{k}-1}M_{d,i;p,q+\alpha;r,j}^{(k_1;k,k_2)}((\ln t^c)^{\frac{p}{k}}). \tag{15}$$

**Proof:** Using (6), (9), (12) and (1); the result follows as

$${}_{a+}^{\;H}I_t^{\alpha;k}\{\xi_0(t)\}=\frac{1}{k\Gamma^{(k)}(\alpha)}\int_a^t(\ln t-\ln w)^{\frac{\alpha}{k}-1}\xi_0(w)\frac{dw}{w}$$

$$=\frac{1}{k\Gamma^{(k)}(\alpha)}\int_a^t(\ln t-\ln w)^{\frac{\alpha}{k}-1}(\ln w)^{\frac{q}{k}-1}M_{d,i;p,q;r,j}^{(k_1;k,k_2)}((\ln w^c)^{\frac{p}{k}})\frac{dw}{w}$$

$$=\frac{1}{k\Gamma^{(k)}(\alpha)}\int_a^t(\ln t-\ln w)^{\frac{\alpha}{k}-1}(\ln w)^{\frac{q}{k}-1}\sum_{m=0}^{\infty}\frac{(d)_{im}^{(k_1)}}{\Gamma^{(k)}(pm+q)}\frac{((c\ln w)^{\frac{p}{k}})^m}{(r)_{jm}^{(k_2)}}\frac{dw}{w}$$

$$=\frac{1}{k\Gamma^{(k)}(\alpha)}\sum_{m=0}^{\infty}\frac{(d)_{im}^{(k_1)}(c)^{\frac{pm}{k}}}{\Gamma^{(k)}(pm+q)}\frac{1}{(r)_{jm}^{(k_2)}}\int_a^t(\ln t-\ln w)^{\frac{\alpha}{k}-1}(\ln w)^{\frac{pm}{k}+\frac{q}{k}-1}\frac{dw}{w}$$

$$=\frac{1}{k\Gamma^{(k)}(\alpha)}\sum_{m=0}^{\infty}\frac{(d)_{im}^{(k_1)}(c)^{\frac{pm}{k}}}{\Gamma^{(k)}(pm+q)}\frac{1}{(r)_{jm}^{(k_2)}}I^{\ln}_{a,\frac{\alpha}{k}-1,\frac{pm+q}{k}-1}(t).$$

For $a=1$; we find that

$${}_{1+}^{H}I_t^{\alpha;k}\{\xi_0(t)\}=\frac{1}{k\Gamma^{(k)}(\alpha)}\sum_{m=0}^{\infty}\frac{(d)_{im}^{(k_1)}(c)^{\frac{pm}{k}}}{\Gamma^{(k)}(pm+q)}\frac{1}{(r)_{jm}^{(k_2)}}k(\ln(t))^{\frac{pm+\alpha+q}{k}+1}B^{(k)}(\alpha,pm+q)$$

$$=\frac{1}{\Gamma^{(k)}(\alpha)}\sum_{m=0}^{\infty}\frac{(d)_{im}^{(k_1)}(c)^{\frac{pm}{k}}}{\Gamma^{(k)}(pm+q)}\frac{1}{(r)_{jm}^{(k_2)}}(\ln t)^{\frac{\alpha+pm+q}{k}-1}B^{(k)}(pm+q,\alpha)$$

$$=\frac{1}{\Gamma^{(k)}(\alpha)}\sum_{m=0}^{\infty}\frac{(d)_{im}^{(k_1)}(c)^{\frac{pm}{k}}}{\Gamma^{(k)}(pm+q)}\frac{1}{(r)_{jm}^{(k_2)}}(\ln t)^{\frac{\alpha+pm+q}{k}-1}\frac{\Gamma^{(k)}(pm+q)\Gamma^{(k)}(\alpha)}{\Gamma^{(k)}(pm+q+\alpha)}$$

$$=(\ln t)^{\frac{\alpha+q}{k}-1}\sum_{m=0}^{\infty}\frac{(d)_{im}^{(k_1)}(c)^{\frac{pm}{k}}}{\Gamma^{(k)}(pm+q+\alpha)}\frac{((\ln t)^{\frac{p}{k}})^m}{(r)_{jm}^{(k_2)}}$$

$$=(\ln t)^{\frac{\alpha+q}{k}-1}\sum_{m=0}^{\infty}\frac{(d)_{im}^{(k_1)}}{\Gamma^{(k)}(pm+q+\alpha)}\frac{((\ln t^c)^{\frac{p}{k}})^m}{(r)_{jm}^{(k_2)}}$$

$$=(\ln t)^{\frac{\alpha+q}{k}-1}M_{d,i;p,q+\alpha;r,j}^{(k_1;k,k_2)}((\ln t^c)^{\frac{p}{k}}).$$

**Theorem 2:**

$${}_{1+}^{H}I_t^{\alpha;k}\{(\ln t)^{\frac{q}{k}-1}{}_{\lambda}M_{d,i;p,q;r,j}^{(k_1,k_3;k_2,k,k_4)}((\ln t^c)^{\frac{p}{k}};\sigma)\}$$

$$=(\ln t)^{\frac{\alpha+q}{k}-1}{}_{\lambda}M_{d,i;p,q+\alpha;r,j}^{(k_1,k_3;k_2,k,k_4)}((\ln t^c)^{\frac{p}{k}};\sigma). \qquad (16)$$

**Proof:** Using (6), (10), (12) and (1); the result follows as

$${}_{a+}^{H}I_t^{\alpha;k}\{\xi_1(t)\}=\frac{1}{k\Gamma^{(k)}(\alpha)}\int_a^t(\ln t-\ln w)^{\frac{\alpha}{k}-1}\xi_1(w)\frac{dw}{w}$$

$$=\frac{1}{k\Gamma^{(k)}(\alpha)}\int_a^t(\ln t-\ln w)^{\frac{\alpha}{k}-1}(\ln w)^{\frac{q}{k}-1}{}_{\lambda}M_{d,i;p,q;r,j}^{(k_1,k_3;k_2,k,k_4)}((\ln w^c)^{\frac{p}{k}};\sigma)\frac{dw}{w}$$

$$=\frac{1}{k\Gamma^{(k)}(\alpha)}\int_a^t(\ln t-\ln w)^{\frac{\alpha}{k}-1}(\ln w)^{\frac{q}{k}-1}\sum_{m=0}^{\infty}\frac{B_{\sigma}^{(k_1)}(\lambda+mi,d-\lambda)}{B^{(k_2)}(\lambda,d-\lambda)}\frac{(d)_{mi}^{(k_3)}}{\Gamma^{(k)}(pm+q)}\frac{((\ln w^c)^{\frac{p}{k}})^m}{(r)_{mj}^{(k_4)}}\frac{dw}{w}$$

$$=\frac{1}{k\Gamma^{(k)}(\alpha)}\sum_{m=0}^{\infty}\frac{B_{\sigma}^{(k_1)}(\lambda+mi,d-\lambda)}{B^{(k_2)}(\lambda,d-\lambda)}\frac{(d)_{mi}^{(k_3)}(c)^{\frac{pm}{k}}}{\Gamma^{(k)}(pm+q)}\frac{1}{(r)_{mj}^{(k_4)}}\int_a^t(\ln t-\ln w)^{\frac{\alpha}{k}-1}(\ln w)^{\frac{pm}{k}+\frac{q}{k}-1}\frac{dw}{w}$$

$$=\frac{1}{k\Gamma^{(k)}(\alpha)}\sum_{m=0}^{\infty}\frac{B_{\sigma}^{(k_1)}(\lambda+mi,d-\lambda)}{B^{(k_2)}(\lambda,d-\lambda)}\frac{(d)_{mi}^{(k_3)}(c)^{\frac{pm}{k}}}{\Gamma^{(k)}(pm+q)}\frac{1}{(r)_{mj}^{(k_4)}}I^{\ln}_{a,\frac{\alpha}{k}-1,\frac{pm+q}{k}-1}(t)$$

For $a=1$; we find that

$${}_{1+}^{H}I_{t}^{\alpha;k}\{\xi_1(t)\}$$

$$=\frac{1}{k\Gamma^{(k)}(\alpha)}\sum_{m=0}^{\infty}\frac{B_{\sigma}^{(k_1)}(\lambda+mi,d-\lambda)}{B^{(k_2)}(\lambda,d-\lambda)}\frac{(d)_{mi}^{(k_3)}(c)^{\frac{pm}{k}}}{\Gamma^{(k)}(pm+q)}\frac{1}{(r)_{mj}^{(k_4)}}k(\ln(t))^{\frac{pm+\alpha+q}{k}+1}B^{(k)}(\alpha,pm+q)$$

$$=\frac{1}{\Gamma^{(k)}(\alpha)}\sum_{m=0}^{\infty}\frac{B_{\sigma}^{(k_1)}(\lambda+mi,d-\lambda)}{B^{(k_2)}(\lambda,d-\lambda)}\frac{(d)_{mi}^{(k_3)}(c)^{\frac{pm}{k}}}{\Gamma^{(k)}(pm+q)}\frac{1}{(r)_{mj}^{(k_4)}}(\ln t)^{\frac{\alpha+pm+q}{k}-1}\frac{\Gamma^{(k)}(pm+q)\Gamma^{(k)}(\alpha)}{\Gamma^{(k)}(pm+q+\alpha)}$$

$$=(\ln t)^{\frac{\alpha+q}{k}-1}\sum_{m=0}^{\infty}\frac{B_{\sigma}^{(k_1)}(\lambda+mi,d-\lambda)}{B^{(k_2)}(\lambda,d-\lambda)}\frac{(d)_{mi}^{(k_3)}}{\Gamma^{(k)}(pm+q+\alpha)}\frac{((\ln t^{c})^{\frac{p}{k}})^{m}}{(r)_{mj}^{(k_4)}}$$

$$=(\ln t)^{\frac{\alpha+q}{k}-1}{}_{\lambda}M_{d,i;p,q+\alpha;r,j}^{(k_1,k_3;k_2,k,k_4)}((\ln t^{c})^{\frac{p}{k}};\sigma).$$

The following results also follow in the same analogous way of that of the above theorems.

**Theorem 3:**

$${}_{1-}^{H}I_{t}^{\alpha;k}\{(\ln t)^{\frac{q}{k}-1}M_{d,i;p,q;r,j}^{(k_1;k,k_2)}((\ln t^{c})^{\frac{p}{k}})\}=(-1)^{\frac{\alpha}{k}}(\ln t)^{\frac{\alpha+q}{k}-1}M_{d,i;p,q+\alpha;r,j}^{(k_1;k,k_2)}((\ln t^{c})^{\frac{p}{k}}).\qquad(17)$$

**Theorem 4:**

$${}_{1-}^{H}I_{t}^{\alpha;k}\{(\ln t)^{\frac{q}{k}-1}{}_{\lambda}M_{d,i;p,q;r,j}^{(k_1,k_3;k_2,k,k_4)}((\ln t^{c})^{\frac{p}{k}};\sigma)\}$$

$$=(-1)^{\frac{\alpha}{k}}(\ln t)^{\frac{\alpha+q}{k}-1}{}_{\lambda}M_{d,i;p,q+\alpha;r,j}^{(k_1,k_3;k_2,k,k_4)}((\ln t^{c})^{\frac{p}{k}};\sigma).\qquad(18)$$

### 4.2 Discussion

Since all the other Mittag-Leffler functions defined in this article are particular cases of (9) and (10), therefore, the results, (15) to (18), follow particularly for the particular cases too.

## 5. CONCLUSION

In the study, using the idea of $k-$ extended special functions (Pochhammer $k-$ symbol, $k-$gamma and $k-$beta functions); an extended form (see (8)) of a generalized form of beta function (see (3)) is introduced. On the basis of which a triple$-k$ extended form (see (9)) and a penta$-k$ extended form (see (10)) of some forms of generalized Mittag-Leffler functions (see (4) and (5)) are introduced. Moreover, the Hadamard $k-$fractional integrals of a function (see (13) and (14)) that involves these extended Mittag-Lefflers functions as a factor, are established. Thus, the article is a contribution in the theory of Fractional Calculus.

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