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# Some New Fractional Integrals And Their Semigroup Properties

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**Abstract.**
*In this work, a number of $k-$fractional integrals are introduced on the basis of some classical fractional integrals like Riemann-Loiuville (RL), Hadamard (H), Kober (K) and Ilyas-Farid (IF) integrals with collaboration of extended $k-$Gamma and $k-$Beta functions. The semigroup and the commutative properties of the newly defined RLK (Riemann-Liouville & Kober), HK, extended HK and IFK type $k-$fractioanl integrals are proved.*

***Keywords:*** *Fractional, integral, derivative, Riemann-Liouville, Erdelyi-Kober, Hadamard.*

## 1. INTRODUCTION

The Fractional Calculus (FC) broadens the notion of classical calculus by focusing on integration and differentiation of non-integer orders. The idea of fractional operators was developed almost at the same time as classical calculus. The concept of fractional derivatives was first mentioned by great philosopher Guillaume de l'Hôpital in a letter to Gottfried Wilhelm Leibniz, one of the founders of calculus, in 1695. In 1730s, Leonhard Euler and Joseph-Louis Lagrange worked on the concept, but it didn't gain much attention. Later on, in 1819, Niels Henrik Abel introduced the idea of fractional integrals (FIs), but his work was not widely recognized. During the region 1820-1830, Siméon Denis Poisson and Augustin-Louis Cauchy made significant contributions to the field. Meanwhile Bernhard Riemann's work on the RL integral laid the foundation for modern fractional calculus in 1858. Mathematicians like Oliver Heaviside, Henri Léon Lebesgue and Paul Lévy made important contributions in the early 20th century. The development of modern fractional calculus began, led by researchers like Stanisław Marcin Ulam, Mark Kac and Enrico Scalas during the phase 1960-1970. Fractional calculus has seen significant growth, with applications in various fields, including physics, engineering, signal processing and more in the current era (Machado et al., 2011; Srivastava, 1989; Das, 2011).

Fractional calculus emerged as a response to the limitations of classical calculus in modeling real-world phenomena. The main reasons for its development are many natural processes, like diffusion, relaxation and oscillations, exhibit non-integer order behavior, which classical calculus couldn't accurately describe. FC can capture the "memory" and non-local effects in systems, where the current state depends on past states or distant locations. It helps to describe fractals and self-similar structures, common in nature, by using non-integer dimensions and extends classical calculus to handle more complex problems, providing a more comprehensive framework for mathematical modeling (Torres & Malinowska, 2012; Sabatier et al., 2007). FC was driven by the need to model real-world problems in fields like physics, engineering and signal processing, where classical calculus was insufficient. By addressing these limitations, FC has become a powerful tool for modeling and analyzing complex systems, enabling new insights and applications across various disciplines (Butzer & Westphal, 2000; Baleanu, 2012; Yang & Zhang, 2022).

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Throghout the discussion; we use $j,k \in Z^{+} \cup \{0\}$ , $D_t^k$ as notation for $\frac{d^k}{dt^k}$ etc, and $\alpha,\beta \in R^{+} \cup \{0\}$, where $R^{+}$ is the notation used for the set of positvie real numbers and the notation $S^k$ is used for $\underbrace{S \times S \times ... \times S}_{k \text{ times}}$ for any set $S$ .

Since

$$D_t^k(t^j) = \begin{cases} \frac{j!}{(j-k)!} t^{j-k}, k \le j, \\ 0, \qquad\qquad k > j, \end{cases} \tag{1}$$

Lacroix explored the idea of extending this differentiation rule to cases where $j$ and $k$ could be a non-integer (fractional) value as

$$D_t^{\beta}(t^{\alpha}) = \begin{cases} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\beta+1)} t^{\alpha-\beta}, \beta \le \alpha, \\ 0, \qquad\qquad \beta > \alpha \end{cases} \tag{2}$$

where the Gamma function $\Gamma(\alpha)$ is a key mathematical tool in FC, providing a way to generalize factorials to non-integer values. It is defined as

$$\Gamma(\alpha) = \int_0^{\infty} e^{-x} x^{\alpha-1} dx, \ \ \alpha > 0. \tag{3}$$

It can be seen that $\Gamma(n+1) = n\Gamma(n)$ .

Particularly, for $(\alpha,\beta) = (1,\frac{1}{2})$ ; Lacroix calculated $D_t^{\frac{1}{2}}(t)$, using (2), as

$$\begin{aligned} D_t^{\frac{1}{2}}(t) &= \frac{\Gamma(1+1)}{\Gamma(1-\frac{1}{2}+1)} t^{1-\frac{1}{2}} = \frac{\Gamma(2)}{\Gamma(\frac{3}{2})} t^{\frac{1}{2}} \\ &= \frac{1}{\Gamma(\frac{1}{2}+1)} t^{\frac{1}{2}} = \frac{1}{\frac{1}{2}\Gamma(\frac{1}{2})} t^{\frac{1}{2}} \\ &= \frac{2}{\sqrt{\pi}} \sqrt{t} = 2\sqrt{\frac{t}{\pi}} \end{aligned}$$

(Podlubny, 1994).

### 1.1Fundamental Theorem of Calculus (FTC)

The Fundamental Theorem of Calculus bridges the gap between differentiation and integration, revealing a deep connection between these two fundamental concepts in calculus.
If $\psi$ is continuous on $[p,q]$ and differentiable on $(p,q)$ , then

$$D_x \int_p^x \psi(t)dt = \psi(x). \tag{4}$$

This relation is termed as $1^{\text{st}}$ FTC.

If $\psi$ is continuous and differentiable function on a closed interval $[p,q]$. Then

$$\int_p^q D_t\psi(t)dt = \psi(q) - \psi(p). \tag{5}$$

This result is known as $2^{\text{nd}}$ FTC.

Let $\psi(x,\alpha)$ be a function such that the integral

$$\Psi(\alpha) = \int_{a(\alpha)}^{b(\alpha)} \psi(x,\alpha)\, dx,$$

are well-defined and differentiable, where the limits of integration, $a(\alpha)$ and $b(\alpha)$, and the integrand, $\psi(x,\alpha)$, are functions of the parameter $\alpha$ . If $\psi(x,\alpha)$ and $\frac{\partial \psi(x,\alpha)}{\partial \alpha}$ are continuous functions of $x$ and $\alpha$ for $a(\alpha) \le \alpha \le b(\alpha)$, $\alpha \in [p,q]$ and $a(\alpha)$ and $b(\alpha)$ must be differentiable with respect to $\alpha$ , then

$$D_{\alpha}(\Psi(\alpha))=\int_{a(\alpha)}^{b(\alpha)}\frac{\partial}{\partial\alpha}(\psi(x,\alpha))\,dx+\psi(b(\alpha),\alpha)D_{\alpha}(b(\alpha))-\psi(a(\alpha),\alpha)D_{\alpha}(a(\alpha)).$$

If we dentote $\int_a^t\int_a^{t_n}\cdots\int_a^{t_3}\int_a^{t_2}\psi(t_1)\,dt_1dt_2\ldots dt_{n-1}dt_n$ by $I_n(\psi)(t)$, then using the Leibniz integral rule and the second fundamental theorem of integral calculus, it is easy to see that

$$I_n(\psi)(t)=\frac{1}{(n-1)!}\int_a^t(t-\kappa)^{n-1}\psi(\kappa)d\kappa, \tag{6}$$

which is referred as the Cauchy Repeated Integal Formula. Likewise one may see that if we dentote $\int_a^t\frac{1}{t_1}\int_a^{t_1}\frac{1}{t_2}\cdots\int_a^{t_{n-2}}\frac{1}{t_{n-1}}\int_a^{t_{n-1}}\frac{1}{t_n}\psi(t_n)\,dt_ndt_{n-1}\ldots dt_2dt_1$ by $I_n^*(\psi)(t)$, then reversing the order of integration, one can easily verify that

$$I_n^*(\psi)(t)=\frac{1}{(n-1)!}\int_a^t(\log(\frac{t}{\kappa}))^{n-1}\psi(\kappa)\frac{1}{\kappa}d\kappa. \tag{7}$$

Following the idea of these $n$-fold integrals (6) and (7), the FIs (8) & (9) and (10) & (11) are defined.

**1.2Definitions (The Classical Fractional Integrals)**

The left and right RL FI of order $\alpha>0$ of a function $\psi(t)$ are defined as

$${}^{RL}_{\lambda+}I_t^{\alpha}\{\psi(t)\}=\frac{1}{\Gamma(\alpha)}\int_{\lambda}^{t}(t-\kappa)^{\alpha-1}\psi(\kappa)d\kappa \tag{8}$$

and

$${}^{RL}_{\mu-}I_t^{\alpha}\{\psi(t)\}=\frac{1}{\Gamma(\alpha)}\int_{t}^{\mu}(\kappa-t)^{\alpha-1}\psi(\kappa)d\kappa, \tag{9}$$

respectively (Farid, 2021).

The left and right Hadamard FI of order $\alpha>0$ of a function $\psi(t)$ are defined as

$${}^{H}_{\lambda+}I_t^{\alpha}\{\psi(t)\}=\frac{1}{\Gamma(\alpha)}\int_{\lambda}^{t}(\ln t-\ln\kappa)^{\alpha-1}\psi(\kappa)\frac{1}{\kappa}d\kappa \tag{10}$$

and

$${}^{H}_{\mu-}I_t^{\alpha}\{\psi(t)\}=\frac{1}{\Gamma(\alpha)}\int_{t}^{\mu}(\ln\kappa-\ln t)^{\alpha-1}\psi(\kappa)\frac{1}{\kappa}d\kappa, \tag{11}$$

respectively.

For a function $\psi(t)$ and for $a>0, b\in\mathbb{R}$, the left and right Kober FI of order $\alpha>0$ are given by

$${}^{K}_{\lambda+}I_{a,b}^{\alpha}\{\psi(t)\}=\frac{a\,t^{-a(b+\alpha)}}{\Gamma(\alpha)}\int_{\lambda}^{t}(t^a-\kappa^a)^{\alpha-1}\kappa^{a(b+1)-1}\psi(\kappa)d\kappa, t>\lambda \tag{12}$$

and

$${}^{K}_{\mu-}I_{a,b}^{\alpha}\{f(t)\}=\frac{a\,t^{ab}}{\Gamma(\alpha)}\int_{t}^{\mu}(\kappa^a-t^a)^{\alpha-1}\kappa^{-a(b+\alpha-1)-1}f(\kappa)d\kappa, t<\mu, \tag{13}$$

respectively (Hanna et al., 2020).

Left and right FI s with expronential functions in the kernel, also known as Ilyas-Farid FIs, of order $\alpha>0$, are defined as

$${}^{IF}_{\lambda+}I_t^{\alpha}\{\psi(t)\}=\frac{1}{\Gamma(\alpha)}\int_{\lambda}^{t}(e^t-e^{\kappa})^{\alpha-1}\psi(\kappa)e^{\kappa}d\kappa,\ t>\lambda \tag{14}$$

and

$${}^{IF}_{\mu-}I_t^{\alpha}\{\psi(t)\}=\frac{1}{\Gamma(\alpha)}\int_{t}^{\mu}(e^{\kappa}-e^t)^{\alpha-1}\psi(\kappa)e^{\kappa}d\kappa, t<\mu, \tag{15}$$

respectively (Ilyas & Farid, 2021).

For $w, z \in \mathbb{C}$ with $\mathrm{Re}(w) > 0$ and $\mathrm{Re}(z) > 0$; the classical Beta function $B(w,z)$ defined by

$$B(w,z) = \int_0^1 \kappa^{w-1}(1-\kappa)^{z-1} d\kappa$$

is connected with the classical Gamma function by the relation

$$B(w,z) = \frac{\Gamma(w)\Gamma(z)}{\Gamma(w+z)}.$$

Let $a < b$ and $\alpha, \beta \in R^+$, Then for any $u, v \in [a,b]$, $u < v$; $\psi : [u,v] \to R$ be a suitable function for which the integral $\int_u^v (\psi(v)-\psi(\tau))^{\alpha}(\psi(\tau)-\psi(u))^{\beta} d\tau$ exists then it is easy to see that

$$\int_u^v (\psi(v)-\psi(\tau))^{\alpha}(\psi(\tau)-\psi(u))^{\beta} d\tau = (\psi(v)-\psi(u))^{\alpha+\beta+1} B(\alpha+1, \beta+1). \tag{16}$$

One may verify it by using the substitution $y = \dfrac{\psi(\tau)-\psi(u)}{\psi(v)-\psi(u)}$. One may observe that the form of (16) for $\psi(z) = z$, $\psi(z) = \log z$ $\psi(z) = e^z$ is responsible for the semigroup proprerty of the RL (defind in Eq. (8)-(9))), Hadamard (in Eqs.(10)-(11)) and the integral defined in equations (14)-(15) FIs, repsectivley.

## 2. LITERATURE REVIEW

Fractional calculus has become a crucial tool across various scientific and engineering disciplines due to its ability to more accurately describe and control systems with complex, non-local and memory-dependent dynamics. Its importance is likely to continue growing as more applications are discovered and as computational techniques continue to improve (Podlubny, 1998; Ross, 2006; Daftardar-Gejji, 2013).

Fractional calculus on a finite interval $[a,b]$ is explored. The work of Hadamard is built and the concepts of integration and differentiation are extended. To establish conditions for boundedness and existence of FIs and derivatives in certain function spaces, including Lebesgue and $L_p$ spaces was the main goal of the paper. It also investigated the semigroup and reciprocal properties of these operators, providing a foundation for further research in FC (Kilbas, 2001).

The integral equation was investigated and conditions for the existence of solutions were established. The equation, which involves a FI operator, was studied on a finite segment of the real line. Explicit formulas for the solutions were derived and properties of the corresponding FIs and derivatives were explored. The existence of solutions was proven in a specific space of functions and the solutions were shown to satisfy certain properties (Kilbas, 2003).

The fundamental concepts of fractional calculus, specifically the RL operators were reviewed. The Taylor-Riemann series was subsequently examined using Osler's theorem, leading to the derivation of double infinite series expansions for certain elementary functions. In the course of this analysis, the convergence of an alternative form of Heaviside's series was proven. A Semi-Taylor series was also introduced as a special case of the Taylor-Riemann series and its connections to special functions were explored through the use of generating functions in complex fractional calculus (Munkhammar, 2004).

A Mikusiński-type operational calculus was developed for a generalized RL fractional differential operator, which encompasses a one-parameter family of fractional derivatives with varying types and orders. The traditional RL and Liouville-Caputo fractional derivatives were shown to be specific cases of this general framework. The constructed operational calculus was then utilized to solve initial-value problems for linear equations involving these generalized fractional derivatives with constant coefficients, yielding solutions for arbitrary orders and types. Specific examples of these solutions were also presented (Hilfer et al., 2009). A novel fractional derivative was established, unifying the RL and Hadamard fractional derivatives into a single, more comprehensive form. Two distinct representations of this generalized derivative were obtained and an illustrative example was provided to demonstrate the results (Katugampola, 2011). Fractional Brownian motion, time-fractional diffusion and standard Brownian motion as special cases, with the Mainardi function emerging as a natural generalization of the Gaussian distribution were inroduced (Pagnini, 2012).

A deeper examination of the RL derivative, a widely employed fractional derivative, was conducted. Additionally, certain previously undiscussed properties of the Caputo derivative were also explored. Furthermore, partial fractional derivatives were introduced, providing valuable insights into the understanding of fractional calculus and its application in modeling diverse phenomena in science and engineering (Li et al., 2011). A generalization of the classical Hadamard FI, which was achieved by utilizing the k-gamma function, was presented. The properties of this extended operator were examined, including its semigroup property, commutative law and boundedness (Farid & Habibullah, 2015).

The semigroup and reciprocal properties of Hadamard-type fractional operators were examined. Conditions for solving certain Hadamard-type fractional differential equations (HTFDEs) were established using the Banach contraction mapping principle. A novel Gronwall inequality with weak singularity was proved and the dependence of HTFDE solutions on derivative order and perturbation terms was analyzed. Illustrative examples were also provided (Ma & Li, 2017).

The densities of products and ratios of independently distributed positive scalar random variables $x_1$ and $x_2$ were examined. Generalizations of Kober operators were explored, including pathway ideas and Gauss' hypergeometric series, offering a broader framework that encompasses various special cases, such as Saigo operators, with statistical interpretations (Mathai & Haubold, 2017). The RL version was deemed most suitable, yet numerical approximations mostly employed the Caputo version. This paper focuses on numerical approximations of fractional differentiation based on the RL definition, covering various kernel types: power-law, generalized Mittag-Leffler-law and exponential-decay-law (Atangana & Gómez-Aguilar, 2018).

A probabilistic interpretation of Kober's fractional integration was proposed, showing that it can be expressed as a constant multiple of an expected value. The associated random variable represents dilation, following a gamma distribution. Similar interpretations were found for EK fractional integration and fractional differential operators (Tarasov & Tarasova, 2019). This research focused on bounding RL FIs using (h-m)-convex functions. Upper bounds for the sum of left and right FIs were established and a modulus inequality was derived using (h-m)-convexity. A Hadamard-type inequality was also obtained with an additional condition. Various special cases of the results were identified (Farid, 2021).

A new Caputo-type modification of the Erdélyi-Kober fractional derivative was introduced. Representations of Erdélyi-Kober FI and derivative operators were formulated. Properties of the new modification and its relationships with other Erdélyi-Kober fractional derivatives were derived. A numerical method was also presented to solve fractional differential equations involving the proposed derivative, with potential for wide application in simulating fractional models (Odibat & Baleanu, 2021).

The investigation of a class of boundary value problems for Hadamard fractional differential equations with integral boundary conditions and disturbance parameters, yielding uniqueness results for positive solutions under weaker conditions was enabled (Liu & Liu, 2022).

## 3. MATERIALS AND METHODS

The $k-$gamma function denoted as $\Gamma_k(\alpha)$, is a generalization of the classical gamma function. For $k>0$, $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha)>0$, $\Gamma_k(\alpha)$ is defined as

$$\Gamma_k(\alpha)=\int_0^\infty e^{\frac{-t^k}{k}} t^{\alpha-1}dt. \tag{17}$$

It is easy to see that

$$\Gamma_k(\alpha+k)=\alpha\Gamma_k(\alpha).$$

Similarly, for $k>0$, $\alpha,\beta \in \mathbb{C}$ with $\mathrm{Re}(\alpha)>0, \mathrm{Re}(\beta)>0$, the $k-$Beta function is defined as

$$B_k(\alpha,\beta)=\frac{1}{k}\int_0^1 t^{\frac{\alpha}{k}-1}(1-t)^{\frac{\beta}{k}-1}dt. \tag{18}$$

The relation between $k-$gamma and $k-$Beta function is given as

$$B_k(\alpha,\beta)=\frac{\Gamma_k(\alpha)\Gamma_k(\beta)}{\Gamma_k(\alpha+\beta)}. \tag{19}$$

It can be seen that

$$B(\frac{\alpha}{k},\frac{\beta}{k})=kB_k(\alpha,\beta). \tag{20}$$

For any $(\alpha,k)\in(R^+)^2$ and $(\lambda,\mu)\in R^2$, the left and the right $k-$FIs of RL type, of order $\alpha>0$ of a function $\psi(t)$ are, respectively, defined as

$$ {}^{RL}_{\lambda+}I_k^{\alpha}\{\psi(t)\} = \frac{1}{k\Gamma_k(\alpha)}\int_{\lambda}^{t}(t-\kappa)^{\frac{\alpha}{k}-1}\psi(\kappa)d\kappa $$

and

$$ {}^{RL}_{\mu-}I_k^{\alpha}\{\psi(t)\} = \frac{1}{k\Gamma_k(\alpha)}\int_{t}^{\mu}(\kappa-t)^{\frac{\alpha}{k}-1}\psi(\kappa)d\kappa $$

(Mubeen & Habibullah, 2012).

For any $(\alpha,k)\in(R^+)^2$ and $(\lambda,\mu)\in R^2$, the left and the right $k-$FI of H-type, of order $\alpha>0$ of a function $\psi(t)$ are, respectively, defined as

$$ {}^{H}_{\lambda+}I_k^{\alpha}\{\psi(t)\} = \frac{1}{k\Gamma_k(\alpha)}\int_{\lambda}^{t}\frac{1}{\kappa}(\log\frac{t}{\kappa})^{\frac{\alpha}{k}-1}\psi(\kappa)d\kappa $$

and

$$ {}^{H}_{\mu-}I_k^{\alpha}\{\psi(t)\} = \frac{1}{k\Gamma_k(\alpha)}\int_{t}^{\mu}\frac{1}{\kappa}(\log\frac{\kappa}{t})^{\frac{\alpha}{k}-1}\psi(\kappa)d\kappa $$

(Farid & Habibullah, 2015).

**Definition 3.1** For any $(\alpha,k)\in(R^+)^2$ and $(a,b,\lambda,\mu)\in R^4$, the left and right $k-$FI of RLK type having order $\alpha>0$ of a function $\psi(t)$ are defined as

$$ {}^{RLK}_{\lambda+}I_{a,b;k}^{\alpha}\{\psi(t)\} = \frac{1}{k\Gamma_k(\alpha)}\int_{\lambda}^{t}(t-\kappa)^{\frac{\alpha}{k}-1}(\frac{\kappa+a}{t+a})^{b}\psi(\kappa)d\kappa,\ t>\lambda \tag{21} $$

and

$$ {}^{RLK}_{\mu-}I_{a,b;k}^{\alpha}\{\psi(t)\} = \frac{1}{k\Gamma_k(\alpha)}\int_{t}^{\mu}(\kappa-t)^{\frac{\alpha}{k}-1}(\frac{\kappa+a}{t+a})^{b}\psi(\kappa)d\kappa,\ t<\mu, \tag{22} $$

respectively.

**Definition 3.2** For any $(\alpha,k)\in(R^+)^2$ and $(a,b,\lambda,\mu)\in R^4$, the left and right HK $k-$FI of order $\alpha>0$ of a function $\psi(t)$ are defined as

$$ {}^{HK}_{\lambda+}I_{a,b;k}^{\alpha}\{\psi(t)\} = \frac{1}{k\Gamma_k(\alpha)}\int_{\lambda}^{t}\frac{1}{\kappa}(\log\frac{t}{\kappa})^{\frac{\alpha}{k}-1}(\frac{\kappa+a}{t+a})^{b}\psi(\kappa)d\kappa,\ t>\lambda \tag{23} $$

and

$$ {}^{HK}_{\mu-}I_{a,b;k}^{\alpha}\{\psi(t)\} = \frac{1}{k\Gamma_k(\alpha)}\int_{t}^{\mu}\frac{1}{\kappa}(\log\frac{\kappa}{t})^{\frac{\alpha}{k}-1}(\frac{\kappa+a}{t+a})^{b}\psi(\kappa)d\kappa,\ t<\mu. \tag{24} $$

**Definition 3.3** For any $(\alpha,k)\in(R^+)^2$ and $(a,b,c,\lambda,\mu)\in R^5$, the left and right extended HK $k-$FI of order $\alpha>0$ are defined as

$$ {}^{HK^*}_{\lambda+}I_{a,b,c;k}^{\alpha}\{\psi(t)\} = \frac{1}{k\Gamma_k(\alpha)}\int_{\lambda}^{t}\frac{1}{\kappa}(\log\frac{t}{\kappa})^{\frac{\alpha}{k}-1}(\frac{\kappa+a}{t+a})^{b}(\log_{\kappa}t)^{c}\psi(\kappa)d\kappa,\ t>\lambda \tag{25} $$

and

$$ {}^{HK^*}_{\mu-}I_{a,b,c;k}^{\alpha}\{\psi(t)\} = \frac{1}{k\Gamma_k(\alpha)}\int_{t}^{\mu}\frac{1}{\kappa}(\log\frac{\kappa}{t})^{\frac{\alpha}{k}-1}(\frac{\kappa+a}{t+a})^{b}(\log_{\kappa}t)^{c}\psi(\kappa)d\kappa,\ t<\mu. \tag{26} $$

**Definition 3.4** For any $(\alpha,k)\in(R^+)^2$ and $(a,b,\lambda,\mu)\in R^4$, the left and right IFK $k-$FI of order $\alpha$ of a function $\psi(t)$ are defined as

$$ {}^{IFK}_{\lambda+}I_{a,b;k}^{\alpha}\{\psi(t)\} = \frac{1}{k\Gamma_k(\alpha)}\int_{\lambda}^{t}(e^{t}-e^{\kappa})^{\frac{\alpha}{k}-1}\left(\frac{e^{\kappa}+a}{e^{t}+a}\right)^{b}e^{\kappa}\psi(\kappa)d\kappa,\ t>\lambda \tag{27} $$

and

$$ {}^{IFK}_{\mu-}I_{a,b;k}^{\alpha}\{\psi(t)\} = \frac{1}{k\Gamma_k(\alpha)}\int_{t}^{\mu}(e^{\kappa}-e^{t})^{\frac{\alpha}{k}-1}\left(\frac{e^{\kappa}+a}{e^{t}+a}\right)^{b}e^{\kappa}\psi(\kappa)d\kappa,\ t<\mu. \tag{28} $$

In this section, some defintions of $k-$FIs of order $\alpha$ are given. In the next section, the related semi-group properties and the commutative properties of the FIs are discussed.

## 4. RESULTS AND DISCUSSIONS

### 4.1. Results

The semigroup property of FC operators (integrals and derivatives) is a fundamental concept in fractional calculus, which enables the use of fractional calculus in various fields, allowing for the modeling and analysis of complex phenomena with non-integer order dynamics. It ensures the applicability of the operators in disciplines of science and engineering like Time-fractional differential equations, Fractional kinetics, Signal processing, Image processing, Control theory, Mathematical finance, Biological systems, Material science, Electrical engineering and Mathematical physics. In this section, we explore the semigroup properties of the RLK, HK, the extended HK and IFK $k-$FIs of order $\alpha$ .

**Theorem 1**

For any $(\alpha,k)\in(R^{+})^{2}$ and $(a,b,\lambda,\mu)\in R^{4}$, we have

$$ {}^{RLK}_{\lambda+}I^{\alpha}_{a,b;k}\{{}^{RLK}_{\lambda+}I^{\beta}_{a,b;k}\{\psi(t)\}\}={}^{RLK}_{\lambda+}I^{\alpha+\beta}_{a,b;k}\{\psi(t)\},t>\lambda $$

and

$$ {}^{RLK}_{\mu-}I^{\alpha}_{a,b;k}\{{}^{RLK}_{\mu-}I^{\beta}_{a,b;k}\{\psi(t)\}\}={}^{RLK}_{\mu-}I^{\alpha+\beta}_{a,b;k}\{\psi(t)\},\ t<\mu. $$

**Proof:**

For $t>\lambda$; using Eq. (21), we find that

$$ \begin{aligned} {}^{RLK}_{\lambda+}I^{\alpha}_{a,b;k}\{{}^{RLK}_{\lambda+}I^{\beta}_{a,b;k}\{\psi(t)\}\} &=\frac{1}{k\Gamma_k(\alpha)}\int_{\lambda}^{t}(t-\kappa)^{\frac{\alpha}{k}-1}\left(\frac{\kappa+a}{t+a}\right)^{b}\,{}^{RLK}_{\lambda+}I^{\beta}_{a,b;k}\{\psi(\kappa)\}\,d\kappa \\ &=\frac{1}{k\Gamma_k(\alpha)}\int_{\lambda}^{t}(t-\kappa)^{\frac{\alpha}{k}-1}\left(\frac{\kappa+a}{t+a}\right)^{b}\frac{1}{k\Gamma_k(\beta)}\int_{\lambda}^{\kappa}(\kappa-\varsigma)^{\frac{\beta}{k}-1}\left(\frac{\varsigma+a}{\kappa+a}\right)^{b}\psi(\varsigma)d\varsigma\,d\kappa \\ &=\frac{1}{k^{2}\Gamma_k(\alpha)\Gamma_k(\beta)}\int_{\lambda}^{t}(t-\kappa)^{\frac{\alpha}{k}-1}\left(\frac{\kappa+a}{t+a}\right)^{b}\int_{\lambda}^{\kappa}(\kappa-\varsigma)^{\frac{\beta}{k}-1}\left(\frac{\varsigma+a}{\kappa+a}\right)^{b}\psi(\varsigma)d\varsigma\,d\kappa \\ &=\frac{1}{k^{2}\Gamma_k(\alpha)\Gamma_k(\beta)}\int_{\lambda}^{t}(t-\kappa)^{\frac{\alpha}{k}-1}\int_{\lambda}^{\kappa}(\kappa-\varsigma)^{\frac{\beta}{k}-1}\left(\frac{\varsigma+a}{t+a}\right)^{b}\psi(\varsigma)d\varsigma\,d\kappa. \end{aligned} $$

Reversing the order of the integrals, it follows that

$$ {}^{RLK}_{\lambda+}I^{\alpha}_{a,b;k}\{{}^{RLK}_{\lambda+}I^{\beta}_{a,b;k}\{\psi(t)\}\}=\frac{1}{k^{2}\Gamma_k(\alpha)\Gamma_k(\beta)}\int_{\lambda}^{t}\psi(\varsigma)\left(\frac{\varsigma+a}{t+a}\right)^{b}\int_{\varsigma}^{t}(t-\kappa)^{\frac{\alpha}{k}-1}(\kappa-\varsigma)^{\frac{\beta}{k}-1}d\kappa\,d\varsigma. $$

Putting Eq. (16) for $\psi(z)=z$, the above equation leads to

$$ {}^{RLK}_{\lambda+}I^{\alpha}_{a,b;k}\{{}^{RLK}_{\lambda+}I^{\beta}_{a,b;k}\{\psi(t)\}\}=\frac{1}{k^{2}\Gamma_k(\alpha)\Gamma_k(\beta)}\int_{\lambda}^{t}\psi(\varsigma)\left(\frac{\varsigma+a}{t+a}\right)^{b}(t-\varsigma)^{\frac{\alpha}{k}+\frac{\beta}{k}-1}B\left(\frac{\alpha}{k},\frac{\beta}{k}\right)d\varsigma. $$

It, by use of (20), takes the form

$$ \begin{aligned} {}^{RLK}_{\lambda+}I^{\alpha}_{a,b;k}\{{}^{RLK}_{\lambda+}I^{\beta}_{a,b;k}\{\psi(t)\}\} &=\frac{1}{k^{2}\Gamma_k(\alpha)\Gamma_k(\beta)}\int_{\lambda}^{t}\psi(\varsigma)\left(\frac{\varsigma+a}{t+a}\right)^{b}(t-\varsigma)^{\frac{\alpha+\beta}{k}-1}kB_k(\alpha,\beta)\,d\varsigma \\ &=\frac{kB_k(\alpha,\beta)}{k^{2}\Gamma_k(\alpha)\Gamma_k(\beta)}\int_{\lambda}^{t}(t-\varsigma)^{\frac{\alpha+\beta}{k}-1}\left(\frac{\varsigma+a}{t+a}\right)^{b}\psi(\varsigma)\,d\varsigma. \end{aligned} $$

With use of Eq. (19), it becomes

$$ \begin{aligned} {}^{RLK}_{\lambda+}I^{\alpha}_{a,b;k}\{{}^{RLK}_{\lambda+}I^{\beta}_{a,b;k}\{\psi(t)\}\} &=\frac{k\frac{\Gamma_k(\alpha)\Gamma_k(\beta)}{\Gamma_k(\alpha+\beta)}}{k^{2}\Gamma_k(\alpha)\Gamma_k(\beta)}\int_{\lambda}^{t}(t-\varsigma)^{\frac{\alpha+\beta}{k}-1}\left(\frac{\varsigma+a}{t+a}\right)^{b}\psi(\varsigma)\,d\varsigma \\ &=\frac{1}{k\Gamma_k(\alpha+\beta)}\int_{\lambda}^{t}(t-\kappa)^{\frac{\alpha+\beta}{k}-1}\left(\frac{\kappa+a}{t+a}\right)^{b}\psi(\kappa)\,d\kappa \\ &={}^{RLK}_{\lambda+}I^{\alpha+\beta}_{a,b;k}\{\psi(t)\}. \end{aligned} $$

Likewise, for $t<\mu$; using Eq. (22), we get

$$ \begin{aligned} {}^{RLK}_{\mu-}I^{\alpha}_{a,b;k}\{{}^{RLK}_{\mu-}I^{\beta}_{a,b;k}\{\psi(t)\}\} &=\frac{1}{k\Gamma_k(\alpha)}\int_{t}^{\mu}(\kappa-t)^{\frac{\alpha}{k}-1}\left(\frac{\kappa+a}{t+a}\right)^{b}\,{}^{RLK}_{\mu-}I^{\beta}_{a,b;k}\{\psi(\kappa)\}\,d\kappa \\ &=\frac{1}{k\Gamma_k(\alpha)}\int_{t}^{\mu}(\kappa-t)^{\frac{\alpha}{k}-1}\left(\frac{\kappa+a}{t+a}\right)^{b}\frac{1}{k\Gamma_k(\beta)}\int_{\kappa}^{\mu}(\varsigma-\kappa)^{\frac{\beta}{k}-1}\left(\frac{\varsigma+a}{\kappa+a}\right)^{b}\psi(\varsigma)d\varsigma\,d\kappa \end{aligned} $$

$$=\frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_t^{\mu}(\kappa-t)^{\frac{\alpha}{k}-1}(\frac{\kappa+a}{t+a})^b\int_{\kappa}^{\mu}(\varsigma-\kappa)^{\frac{\beta}{k}-1}(\frac{\varsigma+a}{\kappa+a})^b\psi(\varsigma)d\varsigma\,d\kappa$$

$$=\frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_t^{\mu}(\kappa-t)^{\frac{\alpha}{k}-1}\int_{\kappa}^{\mu}(\varsigma-\kappa)^{\frac{\beta}{k}-1}(\frac{\varsigma+a}{t+a})^b\psi(\varsigma)d\varsigma\,d\kappa.$$

By interchanging the order of the integrals, we have

$${}^{RLK}_{\mu-}I^{\alpha}_{a,b;k}\{{}^{RLK}_{\mu-}I^{\beta}_{a,b;k}\{\psi(t)\}\}=\frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_t^{\mu}\psi(\varsigma)(\frac{\varsigma+a}{t+a})^b\int_t^{\varsigma}(\kappa-t)^{\frac{\alpha}{k}-1}(\varsigma-\kappa)^{\frac{\beta}{k}-1}d\kappa\,d\varsigma.$$

Using Eq. (16) for $\psi(z)=z$, in above equation, it follows that

$${}^{RLK}_{\mu-}I^{\alpha}_{a,b;k}\{{}^{RLK}_{\mu-}I^{\beta}_{a,b;k}\{\psi(t)\}\}=\frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_t^{\mu}\psi(\varsigma)(\frac{\varsigma+a}{t+a})^b(\varsigma-t)^{\frac{\alpha}{k}+\frac{\beta}{k}-1}B(\frac{\alpha}{k},\frac{\beta}{k})\,d\varsigma.$$

By applying Eq. (20), it leads to

$${}^{RLK}_{\mu-}I^{\alpha}_{a,b;k}\{{}^{RLK}_{\mu-}I^{\beta}_{a,b;k}\{\psi(t)\}\}=\frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_t^{\mu}\psi(\varsigma)(\frac{\varsigma+a}{t+a})^b(\varsigma-t)^{\frac{\alpha+\beta}{k}-1}kB_k(\alpha,\beta)\,d\varsigma$$

$$=\frac{kB_k(\alpha,\beta)}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_t^{\mu}(\varsigma-t)^{\frac{\alpha+\beta}{k}-1}(\frac{\varsigma+a}{t+a})^b\psi(\varsigma)\,d\varsigma\,.$$

It, by the use of (19), becomes

$${}^{RLK}_{\mu-}I^{\alpha}_{a,b;k}\{{}^{RLK}_{\mu-}I^{\beta}_{a,b;k}\{\psi(t)\}\}=\frac{k\frac{\Gamma_k(\alpha)\Gamma_k(\beta)}{\Gamma_k(\alpha+\beta)}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_t^{\mu}(\varsigma-t)^{\frac{\alpha+\beta}{k}-1}(\frac{\varsigma+a}{t+a})^b\psi(\varsigma)\,d\varsigma$$

$$=\frac{1}{k\Gamma_k(\alpha+\beta)}\int_t^{\mu}(\kappa-t)^{\frac{\alpha+\beta}{k}-1}(\frac{\kappa+a}{t+a})^b\psi(\kappa)\,d\kappa$$

$$={}^{RLK}_{\mu-}I^{\alpha+\beta}_{a,b;k}\{\psi(t)\}.$$

**Theorem 2**

For any $(\alpha,k)\in(R^+)^2$ and $(a,b,\lambda,\mu)\in R^4$, we have

$${}^{RLK}_{\lambda+}I^{\alpha}_{a,b;k}\{{}^{RLK}_{\lambda+}I^{\beta}_{a,b;k}\{\psi(t)\}\}={}^{RLK}_{\lambda+}I^{\beta}_{a,b;k}\{{}^{RLK}_{\lambda+}I^{\alpha}_{a,b;k}\{\psi(t)\}\},t>\lambda$$

and

$${}^{RLK}_{\mu-}I^{\alpha}_{a,b;k}\{{}^{RLK}_{\mu-}I^{\beta}_{a,b;k}\{\psi(t)\}\}={}^{RLK}_{\mu-}I^{\beta}_{a,b;k}\{{}^{RLK}_{\mu-}I^{\alpha}_{a,b;k}\{\psi(t)\}\},\ t<\mu.$$

**Proof:**

Since addition is commutative in $R^+$, the above results follow.

In an analogous way of that of the above theorem, one may prove the results given below:

**Theorem 3**

For any $(\alpha,k)\in(R^+)^2$ and $(a,b,\lambda,\mu)\in R^4$, we find that

$${}^{HK}_{\lambda+}I^{\alpha}_{a,b;k}\{{}^{HK}_{\lambda+}I^{\beta}_{a,b;k}\{\psi(t)\}\}={}^{HK}_{\lambda+}I^{\alpha+\beta}_{a,b;k}\{\psi(t)\},\quad t>\lambda$$

and

$${}^{HK}_{\mu-}I^{\alpha}_{a,b;k}\{{}^{HK}_{\mu-}I^{\beta}_{a,b;k}\{\psi(t)\}\}={}^{HK}_{\mu-}I^{\alpha+\beta}_{a,b;k}\{\psi(t)\},\quad t<\mu.$$

**Theorem 4**

For any $(\alpha,k)\in(R^+)^2$ and $(a,b,c,\lambda,\mu)\in R^5$, we find that

$${}^{HK^*}_{\lambda+}I^{\alpha}_{a,b,c;k}\{{}^{HK^*}_{\lambda+}I^{\beta}_{a,b,c;k}\{\psi(t)\}\}={}^{HK^*}_{\lambda+}I^{\alpha+\beta}_{a,b,c;k}\{\psi(t)\},\quad t>\lambda$$

and

$${}^{HK^*}_{\mu-}I^{\alpha}_{a,b,c;k}\{{}^{HK^*}_{\mu-}I^{\beta}_{a,b,c;k}\{\psi(t)\}\}={}^{HK^*}_{\mu-}I^{\alpha+\beta}_{a,b,c;k}\{\psi(t)\},\quad t<\mu.$$

**Theorem 5**

For any $(\alpha,k)\in(R^+)^2$ and $(a,b,\lambda,\mu)\in R^4$, we have

$${}^{IFK}_{\lambda+}I^{\alpha}_{a,b;k}\{{}^{IFK}_{\lambda+}I^{\beta}_{a,b;k}\{\psi(t)\}\}={}^{IFK}_{\lambda+}I^{\alpha+\beta}_{a,b;k}\{\psi(t)\},\quad t>\lambda$$

and

$$ {}^{IFK}_{\mu-}I^{\alpha}_{a,b;k}\{ {}^{IFK}_{\mu-}I^{\beta}_{a,b;k}\{\psi(t)\}\} = {}^{IFK}_{\mu-}I^{\alpha+\beta}_{a,b;k}\{\psi(t)\}, \ \ t<\mu. $$

The same reason of that of Therorem 2 proves the commutative properties of the FIs defined in (23) to (28).

4.2. **Discussion**

In current study, some $k-$FI like RLK, HK, the extended HK and IFK are introduced. As the semigroup properties for the left and right RLK $k-$FI are proved in theorem 1, the semigroup properties of the HK, the extended HK and the IFK $k-$FI, mentioned in Th# (2), (3) and (4), follow in an analogous fashion. From the above discussion, the proposed $k-$FIs of order $\alpha$ can also be generalized as

For any $(\alpha,k)\in(R^{+})^{2}$ and $(a,b,\lambda,\mu)\in R^{4}$; we defind the $g-$ generalized forms of the above proposed $k-$FIs as

$$ {}^{RL}_{\lambda+}I^{\alpha}_{a;g;k}\{\psi(t)\} = \frac{1}{k\Gamma_k(\alpha)}\int_{\lambda}^{t}(t-\kappa)^{\frac{\alpha}{k}-1}(\frac{g(\kappa)}{g(t)})^{a}\psi(\kappa)d\kappa,\ t>\lambda, \tag{29} $$

$$ {}^{RL}_{\mu-}I^{\alpha}_{a;g;k}\{\psi(t)\} = \frac{1}{k\Gamma_k(\alpha)}\int_{t}^{\mu}(\kappa-t)^{\frac{\alpha}{k}-1}(\frac{g(\kappa)}{g(t)})^{a}\psi(\kappa)d\kappa,\ t<\mu, \tag{30} $$

$$ {}^{HK}_{\lambda+}I^{\alpha}_{a;g;k}\{\psi(t)\} = \frac{1}{k\Gamma_k(\alpha)}\int_{\lambda}^{t}\frac{1}{\kappa}(\log\frac{t}{\kappa})^{\frac{\alpha}{k}-1}(\frac{g(\kappa)}{g(t)})^{a}\psi(\kappa)d\kappa,\ t>\lambda, \tag{31} $$

$$ {}^{HK}_{\mu-}I^{\alpha}_{a;g;k}\{\psi(t)\} = \frac{1}{k\Gamma_k(\alpha)}\int_{t}^{\mu}\frac{1}{\kappa}(\log\frac{\kappa}{t})^{\frac{\alpha}{k}-1}(\frac{g(\kappa)}{g(t)})^{a}\psi(\kappa)d\kappa,\ t<\mu, \tag{32} $$

$$ {}^{HK^*}_{\lambda+}I^{\alpha}_{a,b;g;k}\{\psi(t)\} = \frac{1}{k\Gamma_k(\alpha)}\int_{\lambda}^{t}\frac{1}{t}(\log\frac{t}{\kappa})^{\frac{\alpha}{k}-1}(\frac{g(\kappa)}{g(t)})^{a}(\log_{\kappa}t)^{b}\psi(\kappa)d\kappa,\ t>\lambda, \tag{33} $$

$$ {}^{HK^*}_{\mu-}I^{\alpha}_{a,b;g;k}\{\psi(t)\} = \frac{1}{k\Gamma_k(\alpha)}\int_{t}^{\mu}\frac{1}{t}(\log\frac{\kappa}{t})^{\frac{\alpha}{k}-1}(\frac{g(\kappa)}{g(t)})^{a}(\log_{\kappa}t)^{b}\psi(\kappa)d\kappa,\ t<\mu, \tag{34} $$

$$ {}^{IFK}_{\lambda+}I^{\alpha}_{a;g;k}\{\psi(t)\} = \frac{1}{k\Gamma_k(\alpha)}\int_{\lambda}^{t}(e^{t}-e^{\kappa})^{\frac{\alpha}{k}-1}(\frac{g(\kappa)}{g(t)})^{a}e^{\kappa}\psi(\kappa)d\kappa,\ t>\lambda \tag{35} $$

and

$$ {}^{IFK}_{\mu-}I^{\alpha}_{a;g;k}\{\psi(t)\} = \frac{1}{k\Gamma_k(\alpha)}\int_{\lambda}^{t}(e^{t}-e^{\kappa})^{\frac{\alpha}{k}-1}(\frac{g(\kappa)}{g(t)})^{a}e^{\kappa}\psi(\kappa)d\kappa,\ t<\mu. \tag{36} $$

**THE SEMIGROUP AND HENCE THE COMMUTATIVE PROPERTIES OF THE $g-$ GENERALIZED FIS DEFINED IN (29) TO (36) FOLLOW IN THE SAME FASHION OF THAT OF FIS DEFINED IN (21) TO (28).**

## 5. CONCLUSION

In this paper, using the idea of some classical FIs and $k-$extended special funcions (Gamma and Beta), a number of new $k-$FIs (left and right) of the type RLK, HK, the extended HK and IFK are introduced. Moreover, for a given function $\psi(t)$; the $g-$ generalized forms (left and right) of all the newly defined $k-$FIs are also introduced. A special integral representation of the classical Beta function is also proved for a general function $\psi(t)$. Using this special representation, the semigroup properties of all the newly defined FIs are proved. The commutative properties follow directly from the semigroup properties. It is expected that the proposed $k-$FIs will have practical applications in the field of sciences and engineering. Moreover, the boundedness and various other properties of the proposed integrals may also be studied.

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